



*Consumption - **Energy** - Innovation*

The Invisible Hand as an Emergent Property

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THE INVISIBLE HAND AS AN EMERGENT PROPERTY

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ABSTRACT. We develop a multi-sector competitive economy where firms reallocate across sectors under myopic profit-seeking behaviour and quadratic reallocation costs. The dynamic path, formalised as a gradient flow in Wasserstein space, yields two emergent properties: (i) short-run equilibria can be seen as solutions to sequential aggregate optimisation problems; and (ii) the long-run equilibrium is globally stable and efficient. These properties are robust to several extensions, though may fail under fixed reallocation costs. Drawing on EU firm-level data (2018-2023), we find convergence in sectoral profit rates but no labour productivity, with moderate substitutability, small intra-sectoral externalities, and no significant reallocation fixed costs.

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Keywords: out-of-equilibrium dynamics, positive general equilibrium theory, multi-sector economy, myopic firms, firm heterogeneity, intra-industry reallocation, Wasserstein space, gradient flow.

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1. INTRODUCTION

The Walrasian notion of general competitive equilibrium, in its original form (Walras 1874), describes a situation in which prices clear all markets and no forces of disequilibrium remain. The question of how decentralized agents coordinate to reach such an outcome has challenged economists since Walras’s time (see Edgeworth 1881; and, more recently, Hayek 1945; Arrow et al. 1959) and has inspired a rich literature on the dynamics of economic adjustment (e.g., Sonnenschein 1982; Mas-Colell 1986; Artzner et al. 1986).

As emphasized by Stiglitz (2018), this strand of literature argues that economic theories should account for a multisectoral economy characterized by heterogeneous profit rates and wages across sectors, along with a continuous reallocation of firms and workers. Similar concerns are central to the theory of *medium run*, which promotes a framework in which prices, wages, and stocks adjust at different speeds (Blanchard 1997; Solow 2000). In this respect, *common-sense* microfoundations may require a careful reconsideration of how agents form expectations and make decisions in complex environments (Sargent 1993; De Grauwe and Ji 2019; Moll 2024).

A compelling theoretical description of such a reallocation process is provided by Sonnenschein (1982), who characterizes a capitalist economy as a sequence of short-run competitive equilibria. In the short run, “firms are ‘in place’, and the returns to factors that are temporarily immobile are not necessarily equalized. The prices that clear markets at each moment reflect only the relative scarcity of factors that are instantaneously variable”. Over time, all factors become free to move and, guided by simple behavioural rules, “flow toward that branch of production” where returns are higher. This theoretical narrative has rich empirical support and strong predictive power (see, e.g., Simon 1997; Foster et al. 2008).

To delve deeper into these mechanisms, we develop a multi-sector competitive economy in which firms can reallocate across sectors in response to differential returns (Section 2). At each instant, a short-run competitive equilibrium emerges, reflecting sectoral heterogeneity in technology, factor allocation, and consumer preferences (Section 2.1). Sectoral reallocation is governed by myopic profit-seeking behaviour, generating a continuous-time macroeconomic evolution that can be described by a partial differential equation (PDE) (Section 2.2). The possibility of rewriting a PDE as a gradient flow in a suitable space of probability measures,

specifically, the Wasserstein space $\mathcal{W}_2(S)$ (Theorem 2.3), is the key step in transitioning from the dynamics of individual firms to an aggregate representation. This reformulation shows how macroeconomic behaviour emerges from micro-level decisions, revealing an invisible hand mechanism: the uncoordinated pursuit of profit steers the economy along a trajectory of monotonically increasing production and consumption, ultimately converging to an efficient (and Pareto-optimal) allocation (Theorems 2.5 and 2.6)

Our findings appear robust to several extensions, including non-symmetric preferences, intra-sectoral externalities, and immobile labour (Section 2.5 and Appendix B.1.3). The conditions ensuring that the long-run competitive equilibrium is unique and efficient therefore seems fairly general. Notably, the competitive equilibrium may remain efficient in the presence of mild externalities, yet it becomes increasingly inefficient along the dynamic path when strong negative externalities are present (Section 2.5.1). A specific remark deserves the case with a fixed cost of reallocation, where the long-run competitive equilibrium may no longer be unique, and sectoral profit rates can remain persistently unequal (Section 2.5.2).

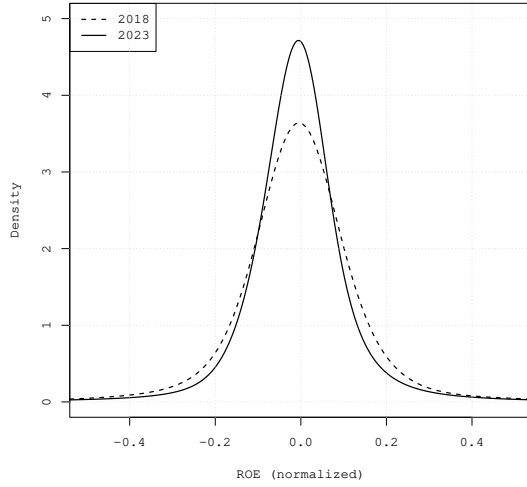
To motivate our framework empirically, Figure 1 plots the cross-sectional distributions, at the sectoral level, of return on equity (ROE) and labour productivity for a balanced panel of EU firms from 2018 to 2023.¹ In Section 4.1, we interpret the convergence in ROE, along with the mild (if any) divergence in labour productivity, as the result of significant firm/capital reallocation across sectors, in contrast to much slower labour reallocation, and it traces the efficiency consequences of this imbalance.

We address questions in four distinct streams of literature, with our main technical innovation being the modelling of firm reallocation dynamics through a partial differential equation (PDE) that admits a gradient flow representation in the *Wasserstein space* $\mathcal{W}_2(S)$, following Artzner et al. (1986).

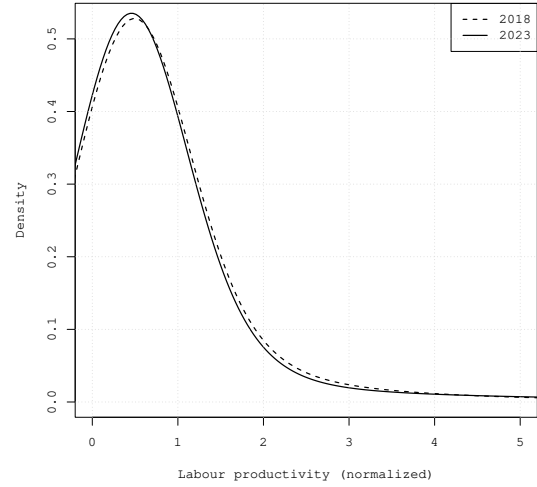
Positive Theory of General Equilibrium. The “positive” theory of general equilibrium, as advanced by Sonnenschein (1981), Sonnenschein (1982), and Artzner et al. (1986), explores economies where firms operate under perfect competition and adjust across sectors based on myopic profit-maximizing rules. This literature grapples with the stability of equilibria, particularly in light of the “impossibility” results of Sonnenschein (1972). Our gradient flow

¹ The sample draws on ORBIS data more than 750,000 firms in 14 EU countries, classified into 680 four-digit NACE sectors.

FIGURE 1. Distribution of (normalized) return on equity (ROE) and labour productivity for 680 four-digit NACE sectors (from 01 to 82) in 2018 and 2023.



(A) The sectoral distribution of ROE (each value divided by the sample mean)



(B) The sectoral distribution of labour productivity (each value divided by the sample mean).

Source: our elaboration on ORBIS data.

representation provides a formal invisible hand result *à la* Smith, where beneficial social and economic outcomes emerge from the self-interested actions of individuals: we demonstrate that the sequence of short-run competitive equilibria corresponds to an instantaneous optimization of a social welfare function. This allows to establish, under general conditions for preferences and technology, the existence, global stability, and the efficiency (maximum production and consumption) of the long-run competitive equilibrium even in the absence of coordinated actions (Artzner et al., 1986; Howitt, 2006) and sophisticated formation of expectations (Simon, 1997).²

Notably, the existence and global stability of the long-run competitive equilibrium are also proved in presence of intersectoral externalities, labour immobility and a fixed cost of reallocation, theoretical aspects which are overlooked in prior studies mainly focused on equilibrium distributions (e.g., Romer, 1986, 1990; Grossman and Helpman, 1991). In this respect,

² More precisely, we define a measure of social utility that is non-decreasing over time and attains its maximum in the long-run competitive equilibrium, where sectoral profit rates are equalized. This shows that local profit information suffices to guide the economy toward efficiency.

we also contribute to the literature on knowledge spillovers (Romer, 1986), and spatial externalities (Fujita and Thisse, 1996; Allen and Arkolakis, 2014) in a general equilibrium setting.

Behavioural Macroeconomics and Agent-Based Decision Rules. The literature advocating simple, empirically grounded decision rules, as discussed by Nelson and Winter (1985), Simon (1997), Kirman (2016), Stiglitz (2018), views the economy as a complex evolving system (Durlauf, 1997) where a multiplicity of agents interact on the basis of quite simple, empirically based, rules. The same spirit, driven by the need to describe a positive theory based on “plausible” behavioural rules for economic agents, underlies the behavioural approach in macroeconomic theory advocated in Sargent (1993), De Grauwe and Ji (2019) and Moll (2024). In these models, agents interact locally and “collective outcomes are ‘emergent properties’ which cannot be imputed to the intention of any single agent” (Dosi and Roventini, 2025). Our framework provides this explicit link: the aggregation of myopic, profit-seeking behaviour is shown to correspond precisely to the optimization of a social welfare function. This demonstrates how decentralized, “plausibly” behavioural rules can lead to an efficient allocation, providing a formal bridge between behavioural micro-foundations and aggregate efficiency. In doing this, we also find that the efficiency of a decentralized equilibrium can be extended also to the presence of externalities and labour immobility.

Firm Reallocation in General Equilibrium Model. The literature on firm reallocation, including works by Dixit and Stiglitz (1977), Maksimovic and Phillips (2001), Melitz (2003), Foster et al. (2008), Bertolotti and Etro (2017), Parenti et al. (2017), and Osotimehin (2019), examines how firms adjust across sectors in response to productivity and demand shocks, often under monopolistic competition. We provide insights into how sectoral heterogeneity in technology, demand, and factor immobility shapes both short- and long-run competitive equilibria. We also show that a fixed cost of reallocation introduces multiple equilibria, with long-run outcomes strongly dependent on initial conditions.

Medium-Run Literature. We finally introduce a new dimension to the medium-run literature, which has traditionally focused on the rigidity of prices and wages, by highlighting the importance of considering the heterogeneity in the timescale associated with the adjustment of different productive factors (Blanchard, 1997; Solow, 2000).

A final remark on why we work with a continuum of differentiated goods. The finite-goods counterpart would not be technically simpler: the corresponding optimization problems would lack interior solutions and continuous dynamics, they would introduce frictions (à la “small menu costs”) that make convergence less smooth and they tend to generate multiple “non-intrinsic” equilibria (since such multiplicity stems from the modeling choice itself; these frictions vanish as the number of goods increases). To our knowledge, the most advanced works using a comparable theoretical framework in the finite case are recent contributions in spatial economics (where sectors are replaced by locations, e.g., [Caliendo et al., 2019](#); [Kleinman et al., 2023](#)). However, these studies focus exclusively on long-run equilibria, without addressing the transitional or out-of-equilibrium dynamics that are central to our paper. Instead, thanks to the continuous structure of the goods space, we can reformulate the problem as a gradient flow in Wasserstein space, which, as said, allows us to describe it as a sequence of short-run competitive equilibria converging to a globally stable long-run equilibrium, and thus to highlight several economic intuitions. In this regard, introducing a continuum of goods is not a stylistic choice: it is part of a methodological innovation, and possibly a first step toward future finite-goods formulations.

The paper is organized as follows. Section [2](#) introduces the model under the assumption of perfect labour mobility, presents its gradient flow representation, and analyses the properties of the long-run competitive equilibrium. It then extends the framework to incorporate heterogeneous preferences, reallocation costs, and intra-sectoral externalities. Section [3](#) examines the implications of labour immobility. Section [4](#) provides a preliminary quantitative assessment of the model, testing its main predictions on firm-level data and presenting numerical illustrations. Section [5](#) concludes and outlines directions for future research. The Appendix collects proofs, supplementary remarks, and additional descriptive material.

2. THE DYNAMICS OF FIRMS DISTRIBUTION

This section presents the core mechanics of the model. We begin by establishing the *short-run competitive equilibrium* through the analysis of consumer and firm decisions (Section [2.1](#)), assuming perfect labour mobility across sectors and instantaneous adjustment of prices and wages. We then introduce firms’ dynamics, namely their reallocation across sectors in

response to profit differentials (Section 2.2). This dynamic process gives rise to a PDE that governs the evolution of the economy, providing the basis for the analysis of the gradient flow representation in Section 2.3. Section 2.4 characterizes the long-run competitive equilibrium, and Section 2.5 discusses several extensions.

2.1. The short-run competitive equilibrium. Inspired by Dixit and Stiglitz (1977), we consider an economy with a continuum of identical consumers, each earning income y from both wages and profits. The total mass of consumers/workers is normalized to $L = 1$, so aggregate income equals $Y = y$. Each consumer is endowed with one unit of labour, supplied inelastically, and holds homogeneous preferences over a continuum of goods of measure n . On the production side, there is a continuum of firms, each differentiated by the good it produces. Let the goods be indexed by $i \in [0, n]$. Since the relations derived in this section apply at every point in time, we omit time indices from the notation for clarity.

2.1.1. The demand side. At each time t the (instantaneous) utility is an increasing function u of the consumption of a composite good X :³

$$(1) \quad U = u(X), \quad \text{with} \quad X := \left(\frac{1}{n} \int_0^n x(i)^{(\sigma-1)/\sigma} di \right)^{\sigma/(\sigma-1)},$$

where $\sigma > 0$ and different from 1, measures the elasticity of substitution among different goods and $x(i)$ is the quantity consumed of good i .

Maximization of the utility of each agent (taking into account the budget constraint $\int_0^n p(i)x(i)di \leq y$) leads to the following (per capita and aggregate) demand for good i :⁴

$$(2) \quad x(i) = \frac{y}{P} \left(\frac{p(i)}{P} \right)^{-\sigma} = \frac{y}{p(i)^\sigma} \frac{1}{P^{1-\sigma}} = \frac{Y}{p(i)^\sigma} \frac{1}{P^{1-\sigma}}.$$

where $p(i)$ is the price of good i and

$$(3) \quad P = \left(\int_0^n p(i)^{1-\sigma} di \right)^{1/(1-\sigma)}$$

is the *price index*.

³ Leisure is not present in the utility function, but it is not difficult to insert it as in Artzner et al. (1986).

⁴ Since demand is linear in income, aggregate demand is unaffected by possible heterogeneity in labour or wealth endowments.

2.1.2. *The supply side.* Let μ be the distribution of firms over the range $[0, n]$, such that $\mu(i)$ denotes the mass of firms producing the good i . We normalize the number of firms, meaning that we assume $\int_0^n \mu(i) di = 1$.⁵

The firms producing good i use the following technology:

$$(4) \quad q(i) = A(i)l(i)^\beta,$$

where $q(i)$ is the quantity of good i produced employing $l(i)$ units of labour, $A(i)$ is a good-specific technological parameter, and $\beta \in (0, 1)$ is the output elasticity of labour. The profits for a (representative) firm producing a good i are given by:

$$(5) \quad \pi(i) = p(i)q(i) - w(i)l(i),$$

where $w(i)$ is the wage paid by the firm. The presence of several firms producing goods i is the main departure from the model of monopolistic competition, where just one firm produces each type of good. Since Equation (4) can be seen as the production function of a firm using one unit of capital, $\pi(i)$ is also the *profit rate* of firms producing good i .

Maximization of profits leads to the labour demand of each firm in sector i :

$$(6) \quad l(i) = \left[\frac{\beta p(i)A(i)}{w(i)} \right]^{1/(1-\beta)}.$$

2.1.3. *The equilibrium in labour and good markets.* We assume the existence of two distinct time scales for the adjustment of variables. In particular, we suppose that the goods and labour markets adjust rapidly compared to the pace at which capital is reallocated across sectors. Hence, at any moment t , markets can be regarded as being in equilibrium for the current distribution $\mu(t)$.⁶ Let us denote this a *short-run competitive equilibrium*. This section is devoted to characterize its properties, while the evolution of μ , i.e. the dynamics of capital reallocation, is postponed to Section 2.2.

⁵ With a slight abuse of notation, we will use the symbol μ to denote both the density of the firm distribution (a non-negative function with unit integral) and its associated probability measure, writing $d\mu(i)$ in place of $\mu(i)di$.

⁶ At the equilibrium, by definition, the level of prices and wages are such that, when the households maximize the utility and the firms maximize, the markets of goods and labour clear.

We assume that workers can move across the production of different goods without cost and instantaneously; therefore, the same wage is paid by all firms:

$$(7) \quad w(i) = w \quad \forall i \in [0, n].$$

Moreover, prices are perfectly flexible in order to guarantee that demand is equal to supply in each sector, i.e.:

$$(8) \quad q(i) = x(i) \quad \forall i \in [0, n].$$

Proposition [2.1](#) characterizes the short-run competitive equilibrium of our economy.

PROPOSITION 2.1 (The short-run competitive equilibrium with perfectly mobile labour)
Set as the numéraire the aggregate value of production Y , i.e. $Y = 1$; then, in the short-run competitive equilibrium:

$$(9) \quad p(i) = \frac{A(i)^{-1} \mu(i)^{-\frac{1}{\sigma(1-\beta)+\beta}}}{\left(\int_0^n [A(i) \mu(i)^{1-\beta}]^{\frac{\sigma-1}{\sigma(1-\beta)+\beta}} di \right)^{1-\beta}};$$

$$(10) \quad \pi(i) = (1 - \beta) \left\{ \frac{A(i)^{\frac{\sigma-1}{\sigma(1-\beta)+\beta}} \mu(i)^{-\frac{1}{\sigma(1-\beta)+\beta}}}{\int_0^n [A(i) \mu(i)^{1-\beta}]^{\frac{\sigma-1}{\sigma(1-\beta)+\beta}} di} \right\};$$

$$(11) \quad P = \left(\int_0^n [A(i) \mu(i)^{1-\beta}]^{\frac{\sigma-1}{\sigma(1-\beta)+\beta}} di \right)^{\frac{\sigma(1-\beta)+\beta}{1-\sigma}};$$

$$(12) \quad w = \beta;$$

and

$$(13) \quad \Pi = (1 - \beta),$$

where $\Pi = \int_0^n \pi(i)\mu(i) di$ is the aggregate (and average) profit (rate). Finally, in sector i , production $Y(i)$, and employment $L(i)$ are equal and given by:

$$Y(i) := p(i)q(i)\mu(i) = \frac{[A(i)\mu(i)^{1-\beta}]^{\frac{\sigma-1}{\sigma(1-\beta)+\beta}}}{\int_0^n [A(i)\mu(i)^{1-\beta}]^{\frac{\sigma-1}{\sigma(1-\beta)+\beta}} di} = l(i)\mu(i) =: L(i).$$

Proof. See Appendix [A](#) □

The choice of the *numéraire* $Y = 1$ is driven by the aim to simplify calculations and follows [Grossman and Helpman \(1991\)](#) (see their Equation (3.8) at page 48). Accordingly, the real wage is $w/P = \beta/P$ and real profits are $\Pi/P = (1 - \beta)/P$. The aggregate real income, which corresponds to the aggregate consumption of the composite good X , is given by:

$$(14) \quad X(t) = \frac{1}{P(t)}.$$

Therefore, the utility of the representative consumer, or its welfare, can be expressed as a decreasing function of the price index $P(t)$:

$$(15) \quad U(t) = u(X(t)) = u\left(\frac{1}{P(t)}\right).$$

Proposition [2.1](#) ensures that labour productivity $Y(i)/L(i)$ equals one in each sector as a result of perfect labour mobility. Consequently, in the short-run competitive equilibrium, the allocation of labour across sectors is efficient. Section [3](#) examines the properties of the equilibrium in the opposite case, where labour is immobile.

2.2. The dynamics of firms across sectors. The firms' search for higher profit rates, i.e. the reallocation of capital across sectors, is the key source of the dynamics. To avoid complications arising from boundary effects in the firm dynamics, assume that the set of goods is distributed along a circle of length n , i.e.:

$$(16) \quad S := \left\{x \in \mathbb{R}^2 : ||x|| = \frac{n}{2\pi}\right\}.$$

The set S corresponds to the interval $[0, n]$ with identified endpoints. For every $i, j \in [0, n]$ their distance, denoted by $|i - j|$ is the arc-length distance on S . Functions defined on S can be seen as one real variable functions which are periodic of period n .

Following Artzner et al. (1986), firms move across sectors in response to differences in current profit rates. In particular, a firm producing a good i moves where profit rates are higher, with velocity v , but this reallocation is costly. The nature of these reallocation costs can vary, and their specific form can have important implications for general equilibrium.⁷ In our framework, these costs are implicit, not reflecting a specific waste or loss of capital, such as psychological costs borne by the entrepreneur due to changing the nature of her business or informational and contractual specificities of the sector where capital is allocated. Instead, we retain the quadratic shape of capital reallocation cost used in the literature (Eisfeldt and Rampini, 2006), i.e. the cost function is assumed to be $c(v) = \frac{1}{2}v^2$. The firm in sector i therefore solves the following maximization problem:

$$(17) \quad \max_v \left[\left(\frac{\partial \pi(t, i)}{\partial i} \right) v - \frac{1}{2}v^2 \right].$$

Problem (17) is the continuous-time (or “infinitesimal”) analogue of the solution of a problem in discrete time in which a firm in sector i at time t chooses a new sector j to maximize:

$$(18) \quad \max_{j \in S} \left[\pi(t, j) - \frac{1}{2}|i - j|^2 \right],$$

where $\pi(t, j)$ denotes the current profits in the sector j , and $\frac{1}{2}|i - j|^2$ is the quadratic cost of reallocating production from i to j . Problem (18) highlights the two key assumptions underlying Problem (17): 1) current profits are used as a proxy for future ones, that is, firms have *static expectations*; and 2) only next-period profits are considered, which can be interpreted as a very high firms’ intertemporal discount rate relative to their rate of reallocation.⁸

The solution to Problem (17) is

$$v = \frac{\partial \pi(t, i)}{\partial i},$$

⁷ Eisfeldt and Rampini (2006, p. 385) refer to “capital illiquidity” as the main source of reallocation cost, and they assume that such cost decreases the stock of capital.

⁸ Krugman (1993) makes a similar assumption in a spatial context. As early as Arrow (1986), it was noted that requiring agents to solve full intertemporal problems with rational expectations places unrealistic informational and computational demands on them. More broadly, several authors argue that excessive sophistication in expectation formation is unrealistic in complex environments (Sargent, 1993; De Grauwe and Ji, 2019; Moll, 2024; Blanchard, 2025). In heterogeneous-agent macro, for instance, Moll (2024, p.19) stresses that rational expectations “unnecessarily complicate computations” and should be replaced, suggesting criteria as computational tractability, empirical consistency, and endogenous beliefs that are met by Equation (17).

which implies that the optimal velocity for a firm in a sector i equals the marginal gain in profits from shifting slightly to the right along the range of goods.

To describe the aggregate dynamics of the economy, we need to translate this micro-level behaviour into an evolution equation for the density of firms, $\mu(t, i)$. To do that we use a general result connecting the reallocation of individual entities (e.g. particles) to the evolution of their collective density, which is formalized in the so-called *continuity equation*.⁹ The latter states that for any population moving with a velocity field $v(t, i)$, the local rate of change in their density, $\partial_t \mu$, equals the negative of the spatial divergence of their flux, i.e.¹⁰

$$(19) \quad \partial_t \mu(t, i) = -\partial_i (\mu(t, i) v(t, i)).$$

Hence, using (19), the equation that governs the evolution of the density of the firms $\mu(t, i)$ is:¹¹

$$(20) \quad \partial_t \mu(t, i) = -\partial_i (\mu(t, i) v(t, i)) = -\partial_i (\mu(t, i) \partial_i \pi(t, i)).$$

Taking the expression for profits from Equation (10), given the initial distribution of firms $\mu(\cdot, 0)$, and considering the boundary condition $\mu(t, 0) = \mu(t, n)$ for every $t \geq 0$, yields to

$$(21) \quad \begin{cases} \partial_t \mu(t, i) = -(1 - \beta) \left[\frac{\partial_i \left(\mu(t, i) \partial_i \left(A(i)^{\frac{\sigma-1}{\sigma(1-\beta)+\beta}} \mu(t, i)^{-\frac{1}{\sigma(1-\beta)+\beta}} \right) \right)}{\int_0^n [A(j) \mu(j)^{1-\beta}]^{\frac{\sigma-1}{\sigma(1-\beta)+\beta}} dj} \right], & (t, i) \in \mathbb{R}^+ \times S \\ \mu(t, 0) = \mu(t, n) \\ \mu(0, i) = \mu_0(i). \end{cases}$$

Theorem 2.2 shows that the dynamics of μ described by partial differential equation (PDE) (21) is well posed.

⁹ Originally developed in the context of fluid dynamics to describe the conservation of mass (Pavliotis, 2014).

¹⁰ See, e.g., Pavliotis (2014) for a formal mathematical explanation.

¹¹ Taking a more general shape for the cost function $c(v) = (c_M/2)v^2$, where we can interpret c_M as a measure of the frictions in the firm reallocation, the continuity equation would become $\partial_t \mu(t, i) = -(1/c_M) \partial_i (\mu(t, i) v(t, i))$.

THEOREM 2.2 (Existence, uniqueness, and boundedness of firms distribution) *Let $\mu_0, A: S \rightarrow \mathbb{R}$ be strictly positive and in $C^2(S)$ ¹². Then PDE (21) has a unique solution $\mu(t, \cdot) \in C([0, +\infty), C^2(S))$ and, there exists $M > 0$ such that, $\forall t \geq 0$*

$$\frac{1}{M} \leq \mu(t, i) \leq M, \quad \forall i \in S.$$

Proof. See Appendix A □

2.3. Gradient flow representation of the competitive equilibrium dynamics. In this section, we recast the dynamics in PDE (21) as a *gradient flow* in the *Wasserstein space* $\mathcal{W}_2(S)$, which enables a direct understanding of how firm-level behaviour influences aggregate variables. In particular, Section 2.3.1 recalls the *Wasserstein distance* of index 2 in S , providing a suitable metric structure to analyse the evolution of $\mu(t)$, highlighting how the reallocation costs of firms are related to the structure of $\mathcal{W}_2(S)$. Section 2.3.2 reformulates the PDE (21) as a gradient flow within $\mathcal{W}_2(S)$, that is, we identify a functional $\mathcal{F}(\mu)$ such that the evolution of $\mu(t)$ follows the direction of steepest descent of $\mathcal{F}$ in $\mathcal{W}_2(S)$. This formulation corresponds to a sequence of period-by-period optimizations in the continuous-time limit, as formalized by the Jordan-Kinderlehrer-Otto (JKO) scheme (Jordan et al., 1998). Then, in Section 2.3.3, we examine the key emergent properties of dynamics. Finally, Section 2.3.4 provides a brief overview of reformulating PDEs as gradient flows and of the JKO scheme.

2.3.1. The Wasserstein space. The Wasserstein space with index 2 on S , denoted by $\mathcal{W}_2(S)$, is defined as the set of probability measures $\mu \in \mathcal{P}(S)$ endowed with the distance¹³

$$(22) \quad d_{W_2}(\mu, \nu) = \left(\inf_{T: T_{\#}\mu = \nu} \int_S |i - T(i)|^2 d\mu(i) \right)^{1/2},$$

¹² S is defined in (16), $C^2(S)$ are real continuous functions v defined on S (with $v(0) = v(n)$, with continuous first and second derivatives).

¹³ In general, it is required that the measures have a finite second moment; however, since the space S is bounded in our analysis, this requirement does not restrict the set $\mathcal{P}(S)$. The Wasserstein-2 distance can also be defined using the *Kantorovich formulation*, which considers all possible transport plans between the two measures. The distance is given by $d_{W_2}(\mu, \nu) = \left(\inf_{\pi \in \Pi(\mu, \nu)} \int_{S \times S} |i - j|^2 d\pi(i, j) \right)^{1/2}$, where $\Pi(\mu, \nu)$ is the set of all joint probability measures on $S \times S$ whose first and second marginals are μ and ν , respectively.

where the infimum is taken over all *transport maps* $T : S \rightarrow S$ such that $T_{\#}\mu = \nu$, meaning that the push forward of μ by T equals ν ¹⁴. An important relationship exists between the Wasserstein-2 distance $d_{W_2}(\mu, \nu)$ and the quadratic cost of reallocating firms across sectors. In this respect, consider the solution of Problem (18), i.e.

$$(23) \quad \tilde{T}(i) = \arg \max_{j \in S} \left[\pi(t, j) - \frac{1}{2}|i - j|^2 \right].$$

$\tilde{T}$ is the map of how the firms' reallocation drives the firms' distribution. The total reallocation costs associated with this reallocation, RC , is equal to:

$$RC = \frac{1}{2} \int_S |i - \tilde{T}(i)|^2 d\mu(i),$$

which, apart from $1/2$, has exactly the same form as the cost to be minimized in the definition of $d_{W_2}(\mu, \nu)$ in (22); hence

$$RC_m(\mu, \nu) = \frac{1}{2} (d_{W_2}(\mu, \nu))^2$$

is exactly the *minimum reallocation cost* in the model from taking firm' distribution ν to μ .

2.3.2. The gradient flow representation. We now use the notions introduced above to rewrite the dynamics of μ in PDE (21). In this new formalism, the distribution of firms across sectors at a given time t , $\mu(t, \cdot)$, is regarded as a point in the space $\mathcal{W}_2(S)$, and its evolution can be described as a gradient flow, that is, a dynamics in which, at each instant, the system moves in the direction of the steepest increase of a suitable functional.¹⁵ The appropriate notions of “gradient” and “direction” are those of $\mathcal{W}_2(S)$, and are briefly introduced in Appendix D, along with some intuition. The gradient flow framework also allows the dynamics to be expressed as a sequence of maximization problems over time, namely, via the JKO scheme. Theorem 2.3 expresses rigorously this formalism.

THEOREM 2.3 (The gradient flow representation and JKO scheme) *Let $\mu_0, A : S \rightarrow \mathbb{R}$ be strictly positive and in $C^2(S)$; then the unique solution of PDE (21) is the unique gradient*

¹⁴ The push forward measure $T_{\#}\mu$ of a measure μ on S by a measurable map $T : S \rightarrow S$ is a measure on S defined for any measurable set $A \subseteq S$ by $(T_{\#}\mu)(A) = \mu(T^{-1}(A))$.

¹⁵ In the mathematical literature (see, e.g., Santambrogio 2015), it is customary to consider the negative of the functional in Equation (24), which results in a minus sign in the gradient flow expression (25). This is purely a notational convention; we adopt the positive form to remain consistent with the economic interpretation.

flow in $\mathcal{W}_2(S)$ of

$$(24) \quad \mathcal{F}(\mu) = \left\{ \frac{(1-\beta)[\sigma(1-\beta) + \beta]}{[\sigma(1-\beta) + \beta] - 1} \right\} \log \left(\int_0^n \left[A(i)^{\frac{1}{1-\beta}} \mu(t, i) \right]^{1 - \frac{1}{\sigma(1-\beta) + \beta}} di \right) = \log X(\mu(t))$$

i.e. [16](#)

$$(25) \quad \frac{d\mu}{dt} = \nabla_{W_2} \mathcal{F}(\mu(t)) = \frac{\nabla_{W_2} X(\mu(t))}{X(\mu(t))}, \quad \mu(0, \cdot) = \mu_0.$$

The trajectory of $\mu(t)$ in [\(25\)](#) can be obtained as the limit for $\Delta t \rightarrow 0$ of the JKO scheme, where

$$(26) \quad \mu(t + \Delta t) = \arg \max_{\mu} \left\{ \mathcal{F}(\mu) - \left(\frac{1}{2\Delta t} \right) d_{W_2}^2(\mu, \mu(t)) \right\} = \arg \max_{\mu} \left\{ \log X(\mu) - \left(\frac{1}{\Delta t} \right) RC_m(\mu, \mu(t)) \right\}.$$

Proof. See Appendix [A](#). □

2.3.3. Emergence of properties from gradient flow representation. The possibility of writing a PDE as a gradient flow is a key step in moving from the dynamics of an individual firm to an aggregate representation, that is, to the emergence of macroeconomic behaviour from micro-level decision-making processes. As a first consequence of Theorem [2.3](#), we find that the value of $\mathcal{F}$ increases along the system's dynamics, as stated in Corollary [2.4](#). Therefore, aggregate consumption is non-decreasing over time.

COROLLARY 2.4 (Non-decreasing aggregate consumption) *Assume the hypotheses of Theorem [2.3](#) hold; the functional $\mathcal{F}$ is increasing along the trajectory of the solution of PDE [\(21\)](#).*

Proof. See Appendix [A](#). □

¹⁶ Appendix [D](#) briefly shows that $\nabla_{W_2} \mathcal{F}(\mu(t))$ can be written in a suitable class of functionals which includes the one defined in [\(24\)](#) as

$$-\nabla \cdot \left(\mu \nabla \frac{\delta \mathcal{F}}{\delta \mu} \right),$$

where $\delta \mathcal{F} / \delta \mu$ is the first variation of $\mathcal{F}$ with respect to μ , i.e. the function such that, for any admissible perturbation ν , satisfies

$$\left. \frac{d}{d\epsilon} \mathcal{F}((1-\epsilon)\mu + \epsilon\nu) \right|_{\epsilon=0} = \int_S \frac{\delta \mathcal{F}}{\delta \mu}(x) d(\nu - \mu)(x).$$

In our model, $\mathcal{F}$ corresponds to the natural logarithm of aggregate consumption, but also reflects aggregate production, profits, and wages, the latter being a constant share of total production. Hence, based on Equation (24), the economy evolves along a path which, when measured in $\mathcal{W}_2(S)$, locally maximizes the instantaneous rate of change of aggregate consumption (and, by extension, production, profits, and wages) at each point in time. This result can be intuitively understood by noting that each firm aims to increase its net profits-seeking the highest possible relative gain given its current profit level and the associated reallocation costs. It is worth emphasizing that, taking a step-by-step perspective, the net aggregate consumption/production

$$X(\mu) - \left(\frac{1}{\Delta t} \right) RC_m(\mu, \mu(t))$$

and thus its dynamic path is inefficient from a static standpoint.¹⁷ However, the JKO scheme (26) offers a crucial insight into the efficiency of the dynamics of the firms' distribution: the evolution follows a sequence of period-by-period optimization problems in which two key forces are at play: (i) the change in the natural logarithm of consumption/profits; and (ii) the reallocation cost induced by the changing distribution of firms. This result is particularly striking because it emerges at the aggregate level, yet remains fully consistent with the logic of individual firm behaviour: each firm seeks the highest possible gain relative to its current profitability, while accounting for the costs associated with reallocation.

Since a key step in the gradient flow approach is the identification of the functional $\mathcal{F}$, a natural question arises: to what extent is $\mathcal{F}$ constrained by the specific functional forms adopted in this paper? In our case, the identification of $\mathcal{F}$ from the original PDE (21) is obtained starting from the results of Iacobelli et al. (2019). We argue that the structure of $\mathcal{F}$ is sufficiently robust to allow for its computation under various extensions of the model, as illustrated in Section 2.5, at least in cases where sectoral profit rates are proportional to a power of the mass of firms in each sector.

¹⁷ In particular, if the economy exhibited static efficiency, its evolution would follow the rule:

$$\mu(t + \Delta t) = \arg \max_{\mu} \left\{ X(\mu) - \left(\frac{1}{\Delta t} \right) RC_m(\mu, \mu(t)) \right\},$$

that is, it would maximize total consumption net of the reallocation cost required to “reach” the most productive sectoral configuration, which clearly differs from the structure implied by the JKO scheme, as described in Theorem 2.3

2.3.4. *PDE reformulated as gradient flow and JKO scheme.* In a more general perspective, there exist several classes of PDEs that can be reformulated as gradient flows. For some of these equations, such as the *heat equation* and the *p-Laplacian equation* (Ambrosio et al., 2021 Chapter 13), one can find a gradient flow formulation in $L^2(\Omega)$ spaces. For other classes of equations (e.g., the Fokker-Planck equation, porous medium equation, advection equation, aggregation equation, Keller-Segel system, and crowd motion equations, see Santambrogio, 2017, Chapter 8, and Ambrosio et al., 2021, Chapter 18), including the PDE studied here, which belongs to the class of (ultra) *fast diffusion equations*, the natural formulation is in $\mathcal{W}_2(S)$. More specifically, the PDEs that can be formulated as gradient flows in $\mathcal{W}_2(S)$ are typically expressed as continuity equations of the form (see Santambrogio, 2017, Chapter 5, and also Example 11.1.2, page 281, in Ambrosio et al., 2005):

$$\frac{\partial \mu}{\partial t} = -\nabla \cdot (\mu v),$$

where μ is the probability density, and the velocity field v is given by:

$$v = -\nabla \left(\frac{\delta \mathcal{F}}{\delta \mu} \right),$$

with $\mathcal{F}$ a functional defined on the space of probability measures, and $\delta \mathcal{F} / \delta \mu$ its first variation (see Footnote 16, at page 15 for the definition).

As regards the possibility of using JKO scheme, it is known to converge for equations formulated as gradient flows in $\mathcal{W}_2(S)$ driven by functionals that are bounded from below, lower semicontinuous, and exhibit some form of convexity (see Ambrosio et al., 2005, Chapter 11.1.3). Standard examples include the heat equation, linear Fokker-Planck equations, and the porous medium equation with $m > 1$. For the *fast diffusion equation* ($0 < m < 1$), a JKO-scheme-based construction of weak solutions via non-local approximations has recently appeared in Carrillo et al. (2024), while existence results via the JKO scheme for weighted ultra-fast diffusion equations ($m < 0$) are obtained in Iacobelli et al. (2019), employing weighted L^1 estimates and careful regularizations. For aggregation-diffusion PDEs with regular interaction potentials, JKO convergence is explicitly established in Huang et al. (2024). On the other hand, there exist several classes of equations, such as certain Keller-Segel systems or models with hard state constraints, that fail to satisfy standard JKO convergence

conditions due to the loss of geodesic convexity or insufficient regularity of the driving functional. In these cases, the gradient flow solution is typically defined as a curve of maximal slope or via regularizations (see, e.g., [Ambrosio et al., 2005](#), Chapter 11).

2.4. The long-run competitive equilibrium. In this section, we characterize the long-run competitive equilibrium, an equilibrium in which all markets clear, consumers and firms maximize utility and profits, and neither firms nor workers have an incentive to reallocate. Theorem [2.5](#) establishes the existence, uniqueness, and global convergence of the economy toward this equilibrium. As shown in its proof, identifying a long-run competitive equilibrium amounts to finding a stationary distribution of the dynamics in PDE [\(21\)](#). Finally, Theorem [2.6](#) demonstrates that the long-run competitive equilibrium is efficient, in the sense that aggregate consumption, production, profits, and wages are maximized.

THEOREM 2.5 (The long-run competitive equilibrium) *Assume that the hypotheses of Theorem [2.2](#) hold. Then the economy admits a unique long-run competitive equilibrium, at which the firms' distribution is given by*

$$(27) \quad \mu^{EQ}(i) = \frac{A(i)^{\sigma-1}}{\int_0^n A(j)^{\sigma-1} dj} \quad \forall i \in S,$$

profits rate by $\pi^{EQ}(i) = 1 - \beta$, and production, employment and consumption, respectively, by

$$Y^{EQ}(i) = L^{EQ}(i) = \frac{A(i)^{\sigma-1}}{\int_0^n A(j)^{\sigma-1} dj} \quad \forall i \in S \text{ and} \quad X^{EQ} = \left(\int_0^n A(j)^{\sigma-1} dj \right)^{\frac{1}{\sigma-1}}.$$

Finally, for any initial condition μ_0 , the unique solution of PDE [\(21\)](#) converges to $\mu^{EQ}(\cdot)$ exponentially fast in $L^2(S)$, i.e.

$$(28) \quad \|\mu(t, \cdot) - \mu^{EQ}(\cdot)\|_{L^2(S)} \leq C_1 e^{-C_2 t}$$

for some $C_1, C_2 > 0$.

Proof. See Appendix [A](#) □

In the long-run competitive equilibrium, profit rates are equalized across sectors at the level $1 - \beta$, and the mass of firms is allocated across sectors according to technological progress and consumer preferences. Specifically, more productive sectors attract a larger share of firms only if the elasticity of substitution among goods exceeds one (Equation (27)), that is, when the substitution effect dominates the income effect. In this equilibrium, the distributions of production Y and employment L mirror that of the firms' allocation; that is, labour productivity is equal to one in each sector. This latter finding is not surprising, given the assumptions of symmetric preferences over goods and perfect labour mobility. Section 3 will show that labour immobility can break this symmetry between the distributions of Y and L . Finally, from PDE (21), the speed of convergence appears to be decreasing in β , although we are unfortunately unable to establish how C_1 and C_2 depend specifically on the model's parameters.

Theorem 2.6 states that, in the long-run competitive equilibrium, consumption (as well as production, wages, and profits) is maximized, implying that the long-run competitive equilibrium is efficient.

THEOREM 2.6 (The efficiency of long-run competitive equilibrium) *Assume the hypothesis of Theorem 2.2 hold. Then for every value of $\sigma \in (0, 1) \cup (1, +\infty)$, (27) is the unique distribution that maximize X in the set of non-negative measurable functions $\mu(i)$ on S such that $\int \mu(i)di = 1$, i.e. X^{EQ} is the maximum attainable consumption.*

Proof. See Appendix A □

To summarize, Theorems 2.5 and 2.6 together provide a comprehensive characterization of the dynamics and their efficiency properties. Starting from any initial distribution, the economy converges to a unique long-run competitive equilibrium distribution μ^{EQ} , which arises from decentralized, myopic, profit-seeking behaviour by firms facing reallocation costs. This equilibrium distribution coincides with the one that maximizes aggregate consumption X , and hence welfare $U(X)$, production, profits, and wages. However, as we will discuss in Sections 2.5.1 and 2.5.2, the presence of intra-sectoral externalities and/or the introduction of a fixed cost of reallocation can partially disrupt this result. The key effects on equilibrium of assuming perfect labour mobility warrant a dedicated discussion in Section 3

2.5. Extensions. In this section, we extend our model in two main directions. First, we analyse the role of intra-sectoral externalities, which amplify or mitigate productivity differences through positive or negative spillovers. Then, we introduce a fixed cost of reallocation, which alters sectoral reallocation dynamics and may give rise to multiple long-run equilibria. A third extension, the case of non-symmetric preferences on goods, is presented and analysed in Appendix [B.1.3](#).

Additional extensions, not explored in detail here for reasons of space, include the incorporation of technological progress, more general preferences, and alternative production technologies. In particular, if technology evolves at a constant rate across sectors, i.e. if $A(i, t) = A(i) \exp(gt)$, then all real variables grow at a rate g , while the qualitative properties of the equilibrium remain unchanged. With respect to preferences, our framework can be extended to accommodate subgroups of goods with varying degrees of substitutability or complementarity. Finally, our results generalize to production technologies that yield a constant factor income share, thereby preserving the model's core equilibrium structure.

2.5.1. Intra-sectoral externalities. In this section, we consider the presence of intra-sectoral technological externalities. In particular, sectoral productivity $A(i, \mu(i))$ takes the form

$$(29) \quad A(i, \mu(i)) = A_0(i) \mu(i)^\eta$$

for some real η and an exogenous function A_0 depending only on i . At the sectoral level, $\eta > 0$ in Equation [\(29\)](#) implies that a firm's productivity (and profitability) benefits from the mass of firms operating within its sector. This is consistent with models of knowledge spillovers in production, as in [Romer \(1986\)](#) (see also [Jovanovic and Rob, 1989](#); [Griliches, 1992](#); [Romer, 1990](#)). Applied in a spatial setting, this mechanism underpins the emergence of industrial districts (Marshallian externalities) in the new economic geography literature (see, e.g., [Allen and Arkolakis, 2014](#)). Conversely, $\eta < 0$ implies that a greater sectoral mass of firms negatively affects individual productivity, which is generally explained by the presence of congestion effects and/or negative pecuniary externalities at the sectoral level ([Fujita and Thisse, 2002](#)). At the aggregate level, $\eta > 0$ (resp. $\eta < 0$) implies increasing (resp. decreasing) returns to scale in production.

The short-run competitive equilibrium described by Proposition 2.1 remains the same, except that $A(i)$ is replaced by $A_0(i)\mu(i)^\eta$ (Appendix B.1 contains all technical details); in equilibrium, profits in sector i are given by:

$$(30) \quad \pi(i) = (1 - \beta) \frac{A_0(i)^{\frac{\sigma-1}{\sigma(1-\beta)+\beta}} \mu(i)^{\frac{\eta(\sigma-1)-1}{\sigma(1-\beta)+\beta}}}{\int_0^n [A_0(j)\mu(j)^{1-\beta+\eta}]^{\frac{\sigma-1}{\sigma(1-\beta)+\beta}} dj}$$

while production and employment are given by:

$$(31) \quad Y(i) = L(i) = \frac{[A_0(i)\mu(i)^{1-\beta+\eta}]^{\frac{\sigma-1}{\sigma(1-\beta)+\beta}}}{\int_0^n [A_0(j)\mu(j)^{1-\beta+\eta}]^{\frac{\sigma-1}{\sigma(1-\beta)+\beta}} dj};$$

with labour productivity still equal to one in each sector. Externalities either exacerbate or smooth production/employment differentials across sectors, depending on their sign (i.e. the sign of η), and on the intensity of goods substitutability, i.e. whether σ is greater or less than 1.

The PDE describing the dynamics of firms' distribution is now:

$$(32) \quad \begin{cases} \partial_t \mu(t, i) = - (1 - \beta) \left[\frac{\partial_i \left(\mu(t, i) \partial_i \left(A_0(i)^{\frac{\sigma-1}{\sigma(1-\beta)+\beta}} \mu(t, i)^{\frac{\eta(\sigma-1)-1}{\sigma(1-\beta)+\beta}} \right) \right)}{\int_0^n [A_0(j)\mu(j)^{1-\beta+\eta}]^{\frac{\sigma-1}{\sigma(1-\beta)+\beta}} dj} \right], & (t, i) \in \mathbb{R}^+ \times S \\ \mu(t, 0) = \mu(t, n) \\ \mu(t, i) = \mu_0(i). \end{cases}$$

In the presence of sectoral externalities, in order to obtain results on the existence, uniqueness, and boundedness of the firms' distribution as in Theorem 2.2 and, consequently, to express PDE (32) as a gradient flow, the exponent of $\mu(i)$ must be negative, i.e.:

$$(33) \quad \eta(\sigma - 1) - 1 < 0;$$

Therefore, if $\sigma > 1$, Condition (33) requires that $\eta \in (-\infty, 1/(\sigma - 1))$; that is, there exists an upper bound on the intensity of positive spatial spillovers. Conversely, if $\sigma \in (0, 1)$, then $\eta \in (-1/(1 - \sigma), \infty)$; in this case, there is an upper bound on the strength of negative externalities. Intuitively, in economies with a high (respectively, low) elasticity of substitution

between goods, i.e. $\sigma \gg 1$ (respectively, σ close to 0), and strong, positive (respectively, negative) externalities, i.e. $\eta \gg 0$ (respectively, $\eta \ll 0$), the inflow of firms attracted by higher profits reinforces existing differentials, leading to an increase rather than a decrease in profit dispersion (see Equation (30)). In a continuous-time setting, this disequilibrium dynamic implies that the PDE does not admit an admissible solution, as the distribution diverges in finite time (Hadamard, 1923).

Once assumed that Condition (33) holds, in presence of externalities the functional $\mathcal{F}(\mu)$ reads as

$$(34) \quad \mathcal{F}(\mu) = \left(\frac{1 - \beta}{1 - \beta + \eta} \right) \log \left(\left\{ \int_0^n [A_0(j) \mu(t, j)^{1-\beta+\eta}]^{\frac{\sigma-1}{\sigma(1-\beta)+\beta}} dj \right\}^{\frac{\sigma(1-\beta)+\beta}{\sigma-1}} \right) = \\ = \left(\frac{1 - \beta}{1 - \beta + \eta} \right) \log X(\mu(t)),$$

while the rest of Theorem 2.3 still holds. In the same respect, we can extend Theorem 2.5 to the presence of intra-sectoral externalities, where now:

$$(35) \quad \mu^{EQ}(i) = Y^{EQ}(i) = L^{EQ}(i) = \frac{A_0(i)^{\frac{\sigma-1}{1-\eta(\sigma-1)}}}{\int_0^n A_0(j)^{\frac{\sigma-1}{1-\eta(\sigma-1)}} dj} \quad \forall i \in S$$

and

$$(36) \quad X^{EQ} = \left(\int_0^n A_0(j)^{\frac{\sigma-1}{1-\eta(\sigma-1)}} dj \right)^{\frac{1-\eta(\sigma-1)}{\sigma-1}}.$$

Equations (35) and (36) illustrate how the presence of externalities affects both the long-run distribution of factors across sectors and the equilibrium levels of consumption. The intuition is that a higher value of η corresponds to greater dispersion in factor allocation, as well as higher levels of consumption and production. More importantly, under Condition (33), the gradient flow associated with the functional (34) highlights a result that is not standard in the literature. Specifically, if $\eta > \beta - 1$, that is, if externalities are positive or at least not excessively negative, $X(t)$ increases along the trajectory toward the unique long-run competitive equilibrium μ^{EQ} defined by Equation (35). Furthermore, it can be shown that μ^{EQ} maximizes $X(t)$ among all feasible distributions, i.e. *First Welfare Theorem* holds (the efficient allocations are also Pareto-optimal in our economy). In other words, the decentralized choices made by firms and workers enable the economy to achieve maximum

efficiency in the long run, even in the presence of positive or sufficiently small negative externalities. This finding contrasts with the conventional view in the literature, according to which the presence of externalities undermines the validity of the First Welfare Theorem (see, e.g., [Mas-Colell et al., 1995](#), Ch. 11). The result hinges on the firms' simple decision rule, which implicitly allows them to "internalize" externalities by using the observed profit rate as their primary decision-making variable. On the opposite, if $\eta < \beta - 1 < 0$, that is, if externalities are sufficiently negative, $X(t)$ decreases along the trajectory toward the unique long-run competitive equilibrium and therefore μ^{EQ} and the corresponding X^{EQ} are not socially optimal. The decentralized equilibrium in this scenario is therefore not socially optimal. The key intuition behind these opposite results is the relationship between $\mu(i)$ and $Y(i)$, which crucially depends on the sign of $1 - \beta + \eta$ (Equation [\(31\)](#)).

2.5.2. Fixed costs in the firms' reallocation. It is plausible that in the real world, a portion of firms' reallocation costs is independent of the "intensity" of reallocation, for example due to significant barriers or indivisibilities in sector entry or exit.¹⁸ In presence of a fixed cost of reallocation $c_0 > 0$, the problem of firm in sector i can be formulated as:

$$\max_v (\partial_i \pi(t, i)) v - c(v),$$

but, unlike in Section [2.2](#), the cost function for reallocation appears as:

$$(37) \quad c(v) = \begin{cases} 0 & \text{if } v = 0 \\ \frac{1}{2}v^2 + c_0 & \text{if } v \neq 0 \end{cases},$$

and therefore the optimal choice is:

$$v(t, i) = \begin{cases} 0 & \text{if } |\partial_i \pi(t, i)| \leq \sqrt{2c_0} \\ \partial_i \pi(t, i) & \text{otherwise.} \end{cases}$$

¹⁸ [Eisfeldt and Rampini \(2006\)](#), p. 385) argue that the empirical observation, namely, that in any given year a substantial share of U.S. firms engage in no capital reallocation, can be explained by the presence of fixed adjustment costs.

Consequently, the PDE describing the dynamics of firms reallocation is:

$$(38) \quad \partial_t \mu(t, i) = \begin{cases} 0 & \text{if } |\partial_i \pi(t, i)| \leq \sqrt{2c_0} \\ -(1 - \beta) \left[\frac{\partial_i \left(\mu(t, i) \partial_i \left(A(i)^{\frac{\sigma-1}{\sigma(1-\beta)+\beta}} \mu(t, i)^{-\frac{1}{\sigma(1-\beta)+\beta}} \right) \right)}{\int_0^n [A(i) \mu(i)^{1-\beta}]^{\frac{\sigma-1}{\sigma(1-\beta)+\beta}} di} \right] & \text{otherwise} \end{cases}$$

The presence of a “discontinuity” in the PDE impedes to have a gradient flow representation (and therefore also the JKO scheme) and therefore to reach general conclusions. However, our intuition is that multiple (typically infinitely many) long-run competitive equilibria exist when the profit distribution is sufficiently close to the long-run competitive equilibrium of the problem without fixed costs. Yet, when fixed costs are large enough, there may exist long-run equilibria whose pointwise distance from the equilibrium without fixed costs is not bounded, as the following example illustrates.

As an example, consider the simplest economy where $A(i) = \gamma(i) = 1$ for all i , $\eta = 0$ and $n = 1$; if $c_0 = 0$, Theorem 2.5 states that the only equilibrium firms’ distribution (also asymptotically stable) for the problem without fixed cost is uniform. Consider instead the case with $c_0 > 0$ and the following firms’ distribution:

$$(39) \quad \mu(i) := \begin{cases} \frac{C_\alpha}{i^\alpha} & \text{for } i \in (0, 1/2] \\ \frac{C_\alpha}{(1-i)^\alpha} & \text{for } i \in (1/2, 1), \end{cases}$$

where $\alpha \in (0, 1)$ and $C_\alpha = (1 - \alpha)/2^\alpha$ in order to have $\int_0^1 \mu(j) dj = 1$. It diverges at $i = 0$ (which is identified with $i = 1$). In Appendix B.1.1 we prove that if $\sigma(1 - \beta) + \beta < \alpha$, and the fixed cost is sufficiently high, i.e.

$$(40) \quad c_0 \geq \left(\frac{1}{2} \right) \left\{ \frac{\alpha(1 - \beta)(1 - \alpha)^{\frac{(\sigma-1)(1-\beta)-1}{\sigma(1-\beta)+\beta}}}{2^{\alpha-1+\frac{\alpha}{\sigma(1-\beta)+\beta}} [(1 - \alpha)(\sigma(1 - \beta) + \beta) + \alpha]} \right\}^2$$

then (39) is the long-run competitive equilibrium $\mu^{EQ}(i)$. The non-uniformity in $\mu^{EQ}(\cdot)$ directly implies inequality in profit rates, causing the equilibrium to be inefficient. The magnitude of this inefficiency is directly related to the heterogeneity of profit rates, i.e. to α .

However, a minor role of fixed costs could be explained by the potentially restrictive assumption of comparing the fixed cost, paid once and for all, with instantaneous profits rather than with the expected stream of future profits generated by operating in the sector. Accounting for future profits introduces an intertemporal dimension to firms' decision-making that is difficult to incorporate explicitly into our framework. However, a plausible justification for using current profits as a reference point is that they may serve as the best available proxy for expected future profit flows. Specifically, consider a discrete-time model with static expectations for profits, i.e. $\pi^a(t, i) = \pi(0, i) \forall t > 0$. Under this assumption, the expected profits in the sector i are given by $\sum_{t=0}^{\infty} (1-r)^t \pi^a(t, i) = \pi(0, i)/r$, where r is the intertemporal discount (real interest) rate. The profits' distribution is stable if, for every i and j ,

$$\left| \frac{\pi(0, i)}{r} - \frac{\pi(0, j)}{r} \right| < |i - j|^2 + c_0,$$

which is implied by the stronger constraint:

$$\max_{i,j} \left| \frac{\pi(0, i)}{r} - \frac{\pi(0, j)}{r} \right| < c_0,$$

i.e.

$$\max_{i,j} |\pi(0, i) - \pi(0, j)| < rc_0.$$

Therefore, from an intertemporal perspective, the fixed reallocation cost c_0 should be scaled by the discount rate r to enable comparison with actual profits, highlighting that fixed costs might be even less relevant for firms' reallocation decisions than initially suggested. Specifically, recalling that c_0 represents the fixed cost of reallocating one unit of capital between sectors, a value of $c_0 = 0.2$ corresponds to a reallocation cost equivalent to 20% in terms of instantaneous profit rate differences (i.e., the alternative sector must offer profit rates at least 20% higher). However, once adjusted by a discount rate $r = 10\%$, the effective threshold considered by firms becomes just $0.2 \times 0.1 = 2\%$. Consequently, profit rate differences smaller than 2% would be insufficient to incentivize capital reallocation, implying that, in long-run competitive equilibrium, profit rates across sectors could differ by up to 2%. To conclude, in Section [4.1](#) we will discuss empirical evidence suggesting that fixed costs play, at most, a minor role in the overall reallocation process, at least at the sectoral level.

3. THE ECONOMY WITH IMMOBILE LABOUR

Figure 1b have shown that the distribution of labour productivity remains non-uniform and strongly persistent over a five-year window, suggesting that sectoral wages do not converge in that timeframe. Meanwhile, a substantial body of literature (e.g., Lee and Wolpin, 2006), documents significant and persistent inter-sectoral *labour* reallocation costs, driven by geographic frictions, institutional rigidities, and skill-mismatch barriers. Both evidence point to a large inertia in labour reallocation. In this section, we therefore examine the polar opposite of the perfectly mobile-labour case, namely an economy with completely immobile labour, in which workers are assumed to remain locked into specific sectors and are unable to respond to wage differentials by switching employment. In other words, in the short-run competitive equilibrium also the sectoral allocation of labour L is assumed to be given.

In particular, let $L(i)$ denote the mass of workers in the sector i , fixed over time, with $L(i) \geq 0$ and $\int_0^n L(i) di = 1$. Wages are therefore heterogeneous across sectors and determined by the equality between supply and demand of labour, as stated in Proposition 3.1 reporting the characteristics of the short-run competitive equilibrium.

PROPOSITION 3.1 (The short-run competitive equilibrium with immobile labour) *Let $L(i)$ be the stock of labour in the sector i and set as the numéraire of economy the aggregate value of production Y , i.e. $Y = 1$; then, in the short-run competitive equilibrium with immobile labour:*

$$(41) \quad w(i) = \beta \left\{ \frac{A_0(i)^{1-\frac{1}{\sigma}} L(i)^{\frac{\sigma(\beta-1)-\beta}{\sigma}} \mu(i)^{(1+\eta-\beta)(\frac{\sigma-1}{\sigma})}}{\int_0^n [A_0(j) L(j)^\beta \mu(j)^{1+\eta-\beta}]^{\frac{\sigma-1}{\sigma}} dj} \right\},$$

$$(42) \quad p(i) = \frac{[A_0(i) L(i)^\beta \mu(i)^{1+\eta-\beta}]^{-\frac{1}{\sigma}}}{\int_0^n [A_0(j) L(j)^\beta \mu(j)^{1+\eta-\beta}]^{\frac{\sigma-1}{\sigma}} dj}$$

$$(43) \quad \pi(i) = (1 - \beta) \left\{ \frac{[A_0(i) L(i)^\beta]^{\frac{\sigma-1}{\sigma}} \mu(i)^{-\frac{(\beta-\eta)(\sigma-1)+1}{\sigma}}}{\int_0^n [A_0(j) L(j)^\beta \mu(j)^{1+\eta-\beta}]^{\frac{\sigma-1}{\sigma}} dj} \right\}$$

$$(44) \quad Y(i) = \frac{[A_0(i)L(i)^\beta \mu(i)^{1+\eta-\beta}]^{\frac{\sigma-1}{\sigma}}}{\int_0^n [A_0(j)L(j)^\beta \mu(j)^{1+\eta-\beta}]^{\frac{\sigma-1}{\sigma}} dj}.$$

$$(45) \quad P = \left(\int_0^n [A_0(i)L(i)^\beta \mu(i)^{1+\eta-\beta}]^{\frac{\sigma-1}{\sigma}} di \right)^{\frac{\sigma}{1-\sigma}}$$

and

$$(46) \quad \Pi = 1 - \beta$$

Proof. See Appendix [C](#). □

Both wages and labour productivity appears now sector specific, with the later given by:

$$(47) \quad \frac{Y(i)}{L(i)} = \frac{A_0(i)^{\frac{\sigma-1}{\sigma}} L(i)^{\frac{\sigma(\beta-1)-\beta}{\sigma}} \mu(i)^{\frac{(1+\eta-\beta)(\sigma-1)}{\sigma}}}{\int_0^n [A_0(j)L(j)^\beta \mu(j)^{1+\eta-\beta}]^{\frac{\sigma-1}{\sigma}} dj},$$

and both results decreasing in $L(i)$ as expected, while the relationship with technology and the mass of firm crucially depends on σ .

Following the same steps of Section [2.2](#), we can get the PDE describing the dynamics of firms' distribution:

$$(48) \quad \partial_t \mu(t, i) = -(1 - \beta) \left\{ \frac{\partial_i \left(\mu(t, i) \partial_i \left([A_0(i)L(i)^\beta]^{\frac{\sigma-1}{\sigma}} \mu(t, i)^{-\frac{(\beta-\eta)(\sigma-1)+1}{\sigma}} \right) \right)}{\int_0^n [A_0(j)L(j)^\beta \mu(t, j)^{1+\eta-\beta}]^{\frac{\sigma-1}{\sigma}} dj} \right\}.$$

Under sectoral externalities and complete labour immobility, ensuring existence, uniqueness, and boundedness of the industry-wide firms' distribution, as established in Theorem [2.2](#) and thus enabling the PDE [\(48\)](#) to be formulated as a gradient flow, requires the exponent of $\mu(i)$ to be negative, i.e.:

$$(49) \quad -(\beta - \eta)(\sigma - 1) - 1 = \eta(\sigma - 1) - \beta(\sigma - 1) - 1 < 0.$$

When no externalities are present, i.e. $\eta = 0$, Condition [\(49\)](#) always holds as in the baseline economy with mobile labour; instead, for $\eta \neq 0$, if $\sigma > 1$, Condition [\(49\)](#) holds when $\eta \in (-\infty, \beta + 1/(\sigma - 1))$, while, if $\sigma \in (0, 1)$, when $\eta \in (\beta + 1/(\sigma - 1), \infty)$. The explanation

for these upper and lower bounds strictly follows the one discussed in Section 2.5.1. Assumed that Condition 49 holds, Theorem 2.3, where the functional $\mathcal{F}(\mu)$ is now given by:

$$(50) \quad \mathcal{F}(\mu) = \left(\frac{1-\beta}{1+\eta-\beta} \right) \log \left(\left\{ \int_0^n [A_0(i)L(i)^\beta \mu(i)^{1+\eta-\beta}]^{\frac{\sigma-1}{\sigma}} di \right\}^{\frac{\sigma}{\sigma-1}} \right) = \\ = \left(\frac{1-\beta}{1+\eta-\beta} \right) \log X(\mu(t));$$

maintains its validity (see Theorem C.2 in Appendix C). At the same time, Theorem 2.5, appropriately reformulated, still holds, and in the long-run competitive equilibrium we have:

$$(51) \quad \mu^{EQ}(i) = Y^{EQ}(i) = \frac{[A_0(i)L(i)^\beta]^{\frac{\sigma-1}{(\beta-\eta)(\sigma-1)+1}}}{\int_0^n [A_0(j)L(j)^\beta]^{\frac{\sigma-1}{(\beta-\eta)(\sigma-1)+1}} dj} \quad \forall i \in S.$$

and

$$(52) \quad \left(\frac{Y(i)}{L(i)} \right)^{EQ} = \frac{A_0(i)^{\frac{\sigma-1}{(\beta-\eta)(\sigma-1)+1}} \cdot L(i)^{\frac{\eta(\sigma-1)-1}{(\beta-\eta)(\sigma-1)+1}}}{\int_0^n [A_0(j)L(j)^\beta]^{\frac{\sigma-1}{(\beta-\eta)(\sigma-1)+1}} dj} \quad \forall i \in S$$

The allocation of labour therefore affect the firms and value of production distribution also in the long run; in particular, $\mu^{EQ}(i)$ and $Y^{EQ}(i)$ are positively (negatively) related to the mass of workers $L(i)$ if $\sigma > 1$ (< 1). The relationship between the labour productivity $\left(\frac{Y(i)}{L(i)} \right)^{EQ}$ (and wages) and $L(i)$ is more complex: if $\sigma < 1$, it is negative; while if $\sigma > 1$, for $\eta \in (1/(\sigma-1), \beta + 1/(\sigma-1))$ it is positive, while it is negative for $\eta \in (-\infty, 1/(\sigma-1))$.

We conclude by noting that labour immobility results in a loss of efficiency. In particular, measuring this loss by the difference of consumption in the long-run competitive equilibrium, Proposition 3.2 states that the economy with immobile labour reaches the most efficient equilibrium when labour is allocated as in the mobile labour economy.

PROPOSITION 3.2 *Let ΔX^{EQ} denote the loss in efficiency caused by labour immobility in the long-run competitive equilibrium, measured by the reduction in consumption, i.e.:*

$$(53) \quad \Delta X^{EQ} \equiv \left(\int_0^n A_0(i)^{\frac{\sigma-1}{1-\eta(\sigma-1)}} di \right)^{\frac{1-\eta(\sigma-1)}{\sigma-1}} - \left(\int_0^n [A_0(i)L(i)^\beta]^{\frac{\sigma-1}{1+(\beta-\eta)(\sigma-1)}} di \right)^{\frac{1+(\beta-\eta)(\sigma-1)}{\sigma-1}}.$$

Then, ΔX^{EQ} is always non-negative, with a minimum value of 0 corresponding to the labour allocation of the mobile labour economy.

Proof. See Appendix C. □

Taken together, Theorem 2.6 and Proposition 3.2 suggest that the most efficient factor allocation (of capital and labour) is reached in the long-run competitive equilibrium of the mobile economy.

4. A PRIMER QUANTITATIVE EVALUATION OF THE MODEL

In Section 4.1, we use a large sample of firms to estimate the model's parameters. These parameter values are then employed in Section 4.2 to conduct some numerical experiments.

4.1. Back to the data. This section presents an empirical evaluation of the model. We analyse a sample of over 750,000 firms from 14 EU countries, drawn from the ORBIS dataset for 2018 and 2023, and aggregated into 680 four-digit NACE sectors (ranging from 01 to 82). In particular, Section 4.1.1 provides empirical evidence supporting the fundamental mechanism behind the reallocation of firms across sectors, specifically the convergence dynamics of sectoral profit rates, as measured by *returns on equity* (ROE) between 2018 and 2023. Moreover, it suggests that any potential fixed costs of reallocation do not have a significant impact on the overall dynamics. Section 4.1.3, on the other hand, discusses a method for assessing the degree of labour mobility, concluding that, at least within a 5-year horizon, labour appears to be relatively immobile. Taking the extreme case of labour immobility, we then proceed to calculate the model parameters. A significant limitation of the ORBIS data is the challenge of obtaining a reliable proxy for $\mu(i)$, the stock of physical capital allocated to sector i .¹⁹ To address this issue, in the empirical analysis we focus on ROE as a proxy for profit rates π , the *value of production* as a proxy for Y , and *employment* as a proxy for L .

4.1.1. Convergence in the sectoral profit rates. Theorem 2.5, one of our key results, demonstrates the global convergence to the long-run competitive equilibrium, where sectoral profit rates are equalized. This result holds even in the presence of non-symmetric preferences, intra-sectoral externalities and labour immobility, but does not apply when there are fixed

¹⁹ The only variable in the ORBIS dataset that could potentially serve as a proxy for μ is the total assets reported on balance sheets; however, this includes both physical capital and financial assets, the mix of which is highly heterogeneous across sectors. Since the empirical analysis is not central to the scope of this paper, and for the sake of brevity, we discuss in Appendix F a procedure for estimating $\mu(i)$, based on some assumptions regarding the distribution of $A(i)$.

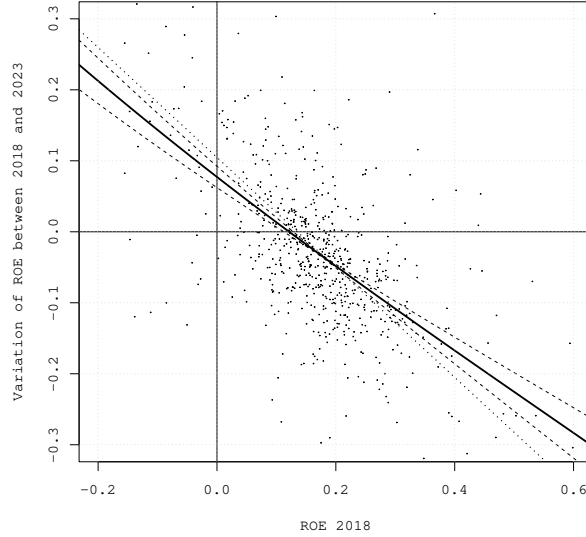
costs of reallocation. A suitable (econometric) model for the dynamics of profit rates is, therefore, the following:

(54)

$$\pi(i, t) - \pi(i, 0) \equiv \Delta\pi(i, t) = m(\pi(i, 0)) + \epsilon(i), \text{ with } m' < 0, m(-\infty) > 0 \text{ and } m(\infty) < 0,$$

and $\epsilon(i)$ is a random disturbance term, assumed to be i.i.d. with zero mean and finite variance. Model (54) implies a global monotonic convergence to a unique long-term equilibrium profit rate $\pi(i)^{EQ} = \bar{\pi}^{EQ}$ defined by $m(\bar{\pi}^{EQ}) = 0$. Theorem 2.5, moreover, states that $\beta = 1 - \bar{\pi}^{EQ}$. Figure 2 reports the Nadaraya-Watson estimation of $m(\cdot)$ of Model (54) with its 95% confidence bands (dashed lines), where profit rate $\pi(i)$ is proxied by the average ROE in the sector i . The estimation suggests the existence of a linear relationship, the estimate

FIGURE 2. The change of ROE between 2018 and 2023 versus ROE in 2018 for 680 four-digit NACE sectors (from 01 to 82). The ticked line is a nonparametric estimate by the Nadaraya-Watson estimation with its 95% confidence bands (dashed lines). The dotted line is the linear regression reported in Table 1.



Source: our elaboration on ORBIS data.

of which is, in turn, given by:²⁰

$$(55) \quad \Delta\text{ROE}(i, 2023) = \underset{(0.008)}{0.105} + \underset{(0.023)}{-0.775}\text{ROE}(i, 2018);$$

²⁰ Standard errors are reported in parentheses. Table 1 in Appendix E reports the standard estimation diagnostics.

the outcome of this estimate is reported by a dotted line in Figure 2. From (55) we have an estimate of $\bar{\pi}^* = 0.105/0.775 \approx 0.14$, i.e. of $\beta = 0.86$. A profit rate of 14% is plausible taking into account that is *nominal* (in that period inflation rate was about 2% in EU) and is *net of depreciation rate* of physical capital. A share of labour on total production equal to 0.86 would be instead overestimated, being in EU in 2022 the ratio between domestic production and compensation of employees around 0.24.²¹ However, we should consider that in our framework two key factors are missing and both inflate the estimate of β , the *depreciation rate* of physical capital, which amounts to about 10% (Gupta et al., 2014), and the *intermediate consumption*, which absorbs about 54% of domestic production in EU in 2022.²² Given a production including the intermediate consumption c as $q = c^\alpha l^\beta$ (the unit of capital is set to one), and assuming that the shares of each input are decided by their marginal productivity, we have that the net profit rate is given by $(1 - \alpha - \beta) - \delta$. where δ is the depreciation rate. Taken $\alpha = 0.54$, $\delta = 0.1$ and a net profit rate in equilibrium equal to 0.14, then $\beta = 0.22$, which is very close to 0.24. The presence of intermediate production does not affect our theoretical results, as it appears as a rescaling of total production. The same holds for the depreciation rate, which does not affect the differential sectoral profit rates. Finally, the smoothness of convergence path to the long-run competitive equilibrium in Figure 2 suggests that the possible presence of fixed costs of reallocation has not significant impact on the sectoral firm reallocation, at least at aggregate level.

4.1.2. *Divergence in the sectoral labour productivity.* When we focus on labour productivity, Figure 3 reveals a divergent trend, i.e. labour productivity across sectors diverges over time, rather than regressing to the mean. Given the close relationship between productivity and wages, this lack of convergence suggests that labour reallocation, which wage differentials should primarily drive, is either absent or proceeds at a much slower pace than capital reallocation. The estimated of the linear econometric model of convergence in labour productivity, i.e.:²³

$$(56) \quad \Delta \log(\text{Labour Productivity}(i, 2023)) = 0.027 + 0.003 \text{Labour Productivity}(i, 2018)$$

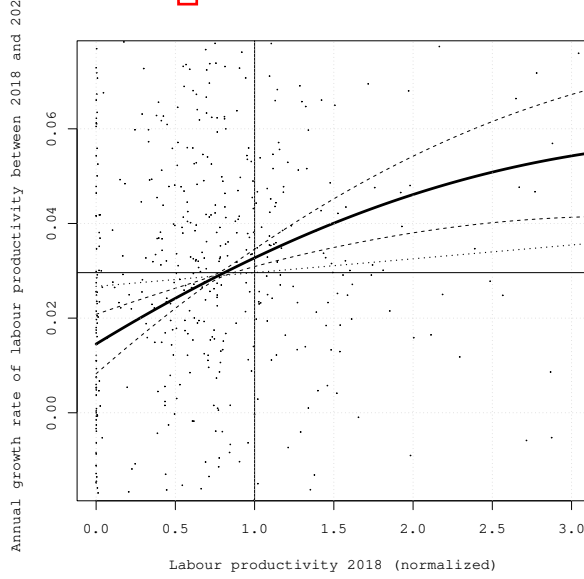
(0.003) (0.001)

²¹ See https://ec.europa.eu/eurostat/statistics-explained/index.php?title=Supply_and_use_tables_for_the_European_Union_and_the_euro_area.

²² As in footnote 21.

²³ Table 2 in Appendix E reports the standard estimation diagnostics.

FIGURE 3. The change of labour productivity between 2018 and 2023 versus labour productivity in 2018 for 680 four-digit NACE sectors (from 01 to 82). The ticked line is a nonparametric estimate by the Nadaraya-Watson estimation with its 95% confidence bands (dashed lines). The dotted line is the linear regression reported in Table 2.



Source: our elaboration on ORBIS data.

confirms the (marginally significant) divergence in sectoral labour productivity.

4.1.3. *Labour mobility and the calibration of model parameters.* Taken together, the evidence to date indicates that the immobile-labour model provides the closest approximation for our sample of firms. A direct comparison of Propositions 2.1 and 3.1 makes clear that the critical difference between the two models lies in the functional relationship between sector-level value of production and employment. With perfect mobile labour, the value of production $Y(i)$ and employment $L(i)$ are the same and profit rate does not play any role; on the contrary, with labour immobility, assuming $\eta = 0$, the relationship is less than proportional, to become indeterminate when $\eta \neq 0$. In particular, from Equation (114) in Appendix C, we get:

$$(57) \quad \log \left(\frac{Y(i, t)}{Y(i, 0)} \right) = g(i) + \left[\frac{\beta(\sigma - 1)}{1 - (\eta - \beta)(\sigma - 1)} \right] \log \left(\frac{L(i, t)}{L(i, 0)} \right) + \left[\frac{(1 + \eta - \beta)(\sigma - 1)}{1 - (\eta - \beta)(\sigma - 1)} \right] \log \left(\frac{\pi(i, t)}{\pi(i, 0)} \right),$$

where $g(i)$ reflects the time change both of $A(i)$ and P .²⁴ The estimate of the corresponding econometric model of Equation (57) is:²⁵

$$(58) \quad \log \left(\frac{Y(i, t)}{Y(i, 0)} \right) = \underset{(0.011)}{0.243} + \underset{(0.027)}{0.214} \log \left(\frac{L(i, t)}{L(i, 0)} \right) + \underset{(0.012)}{0.040} \log \left(\frac{\text{ROE}(i, t)}{\text{ROE}(i, 0)} \right),$$

where the random disturbance of the econometric model takes the possible (exogenous) heterogeneity in $g(i)$, while any other aggregate change is in the estimated constant.

The estimate in (58) point to a some degree of labour immobility, given the low elasticity of the value of production to employment (should be 1 in the case of perfect mobility) and the significant elasticity to profit rates (should be 0 in the case of perfect mobility). Assuming the extreme scenario of labour immobility, the estimate in (58), together with their theoretical counterparts in Equation (57), allow calculating the values of η and σ , i.e. $\eta \approx 0.0222$ and $\sigma \approx 1.3144$ taking into account a value of $\beta = 0.86$ estimated in Section 4.1.1. Therefore, our economy appears to be characterized by positive but small intra-sectoral externalities, while goods and services are moderately substitutable for one another.

4.2. Numerical exploration. In this section, we present a series of numerical explorations, using as a baseline the labour immobile economy, with values of $\beta = 0.86$, $\sigma = 1.3144$, and $\eta = 0.0222$, as calculated in Section 4.1. We assume that $L(i) = [\sin(2\pi(i - 1/4)) + 3]/2$ (in short, the labour distribution on S presents a peak at $i = 0.5$), no heterogeneity in technological progress, i.e., $A(i) = 1 \quad \forall i \in S$, and symmetric preferences. Furthermore, we assume $n = 1$, meaning that the circle S , which represents the variety space of goods, has unit length. Given our assumptions, in the long-run competitive equilibrium, the profit rate $\pi^{EQ}(i) \equiv 0.14$, while the firms' density $\mu^{EQ}(i) = 1 \quad \forall i \in S$.

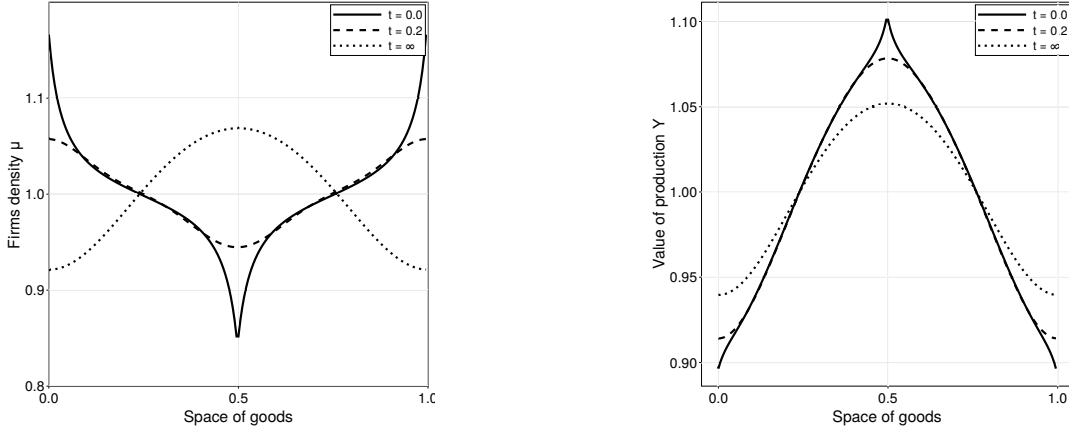
4.2.1. The dynamics of baseline economy. Figures 4a and 4b display the dynamics of firms' distribution and the value of production in our baseline economy with immobile labour, where firms' distribution evolves according to PDE (48). The x -axis represents the space of goods, while the y -axis reports firm density μ and value of production Y . Since sectors are arranged on a circle, the values at positions 0 and 1 coincide. The initial firms' distribution at $t = 0$

²⁴ Equation (57) is robust to the presence of heterogeneous preferences. The expression for $g(i)$ in this case takes into account also the possible time change of the factor $\gamma(i)$.

²⁵ Table 3 in Appendix E reports the standard diagnostics of the estimate.

features a minimum density at position 0.5, mirroring the inverse of the labour distribution. The distribution of value of production in Figure 4b more closely resembles the labour distribution (Proposition 3.1). The evolution of firm density, driven by the distribution of profit rates (Figure 5a), appears non-linear across certain types of goods. In some sectors, the mass of firms initially increases (or decreases) before reversing direction, ultimately resulting in a decline (or rise). As expected, in the long-run competitive equilibrium, the distribution of firms tends to adjust to the distribution of labour.

FIGURE 4. The dynamics of firms and production in the baseline economy

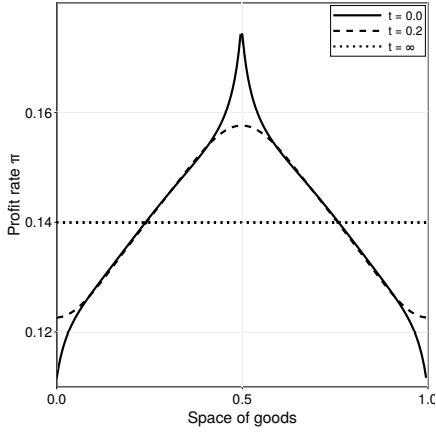


(A) The convergence to the long-run competitive equilibrium of the firms' distribution. On x -axis are reported the space of goods, while on y -axis the firm density.

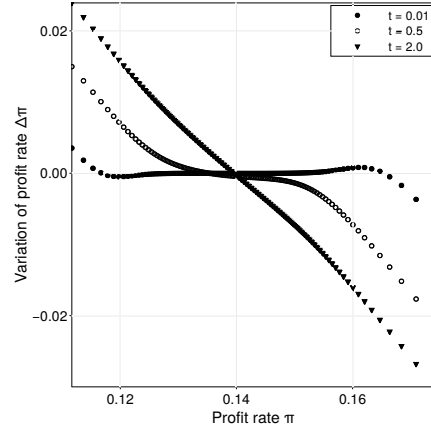
(B) The convergence to the long-run competitive equilibrium of the production distribution. On x -axis are reported the space of goods, while on y -axis the value of production.

Turning our attention to profit rates, these converge non-monotonically towards a uniform equilibrium of $\pi(i) = 0.14 \forall i \in S$, as expected (Figure 5a). These short-run non-linear dynamics are also evident in the relationship between the initial profit rate and its subsequent change over time across different intervals, as illustrated in Figure 5b. However, when observed over a sufficiently long time horizon, the relationship becomes linear, resembling the pattern estimated in Figure 2. By contrast, the distribution of labour productivity remains relatively stable over time (Figure 5c), yet exhibits a clear divergent pattern similar to that observed in our firm-level sample (compare Figures 3 and 5d).

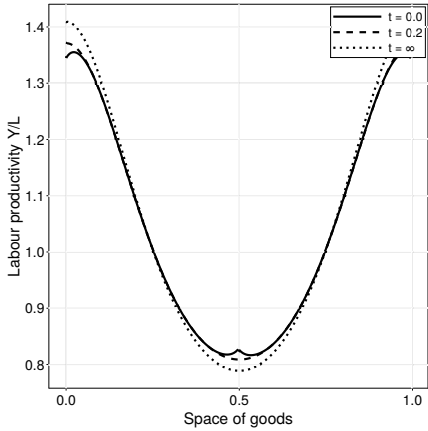
FIGURE 5. The dynamics of profit rates and labour productivity in the base-line economy



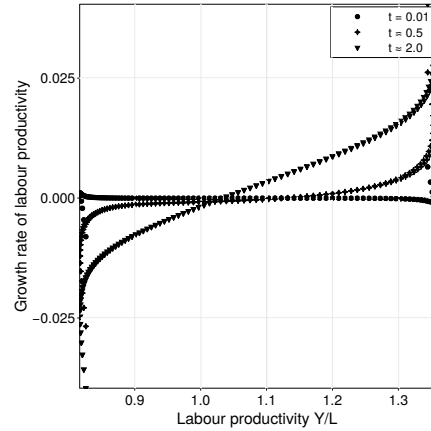
(A) The convergence to the long-run competitive equilibrium of the profit rate distribution. On x -axis are reported the space of goods, while on y -axis the profit rates.



(B) The relationship between the sectoral profit rates at the time 0 (x -axis) and their change over time for different time intervals (0.01, 0.5, 2) of the same (y -axis).



(C) The convergence to the long-run competitive equilibrium of the labour productivity distribution. On x -axis are reported the space of goods, while on y -axis the profit rates.

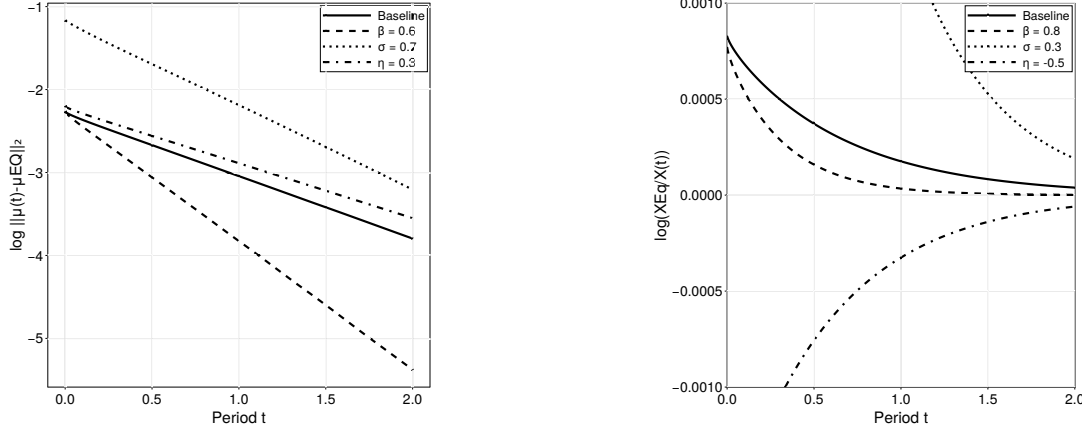


(D) The relationship between the sectoral labour productivities at the time 0 (x -axis) and their change over time of the same (y -axis).

4.2.2. *The convergence to the long-run competitive equilibrium.* Figure 6a illustrates the varying speed of convergence to the long-run distribution of firms, μ^{EQ} , as defined in Equation (51). This also implicitly confirms that the economy is converging toward the long-run competitive equilibrium. The x -axis reports time periods, while the y -axis shows the L^2

distance, which serves as a summary statistic indicating how far the economy is from equilibrium at each point in time.²⁶ The convergence pattern appears to be exponential over time. It is faster for lower values of β and slower for higher values of η .

FIGURE 6. The convergence to the long-run competitive equilibrium for different configuration of parameters.



(A) The speed of convergence to the long-run competitive equilibrium for different values of the parameters β , σ , and η , relative to the baseline economy. The x -axis reports the time period, while the y -axis shows the logarithm of the L^2 distance, defined as the square root of the integral of the squared differences between the current distribution of firms and the long-run competitive equilibrium distribution.

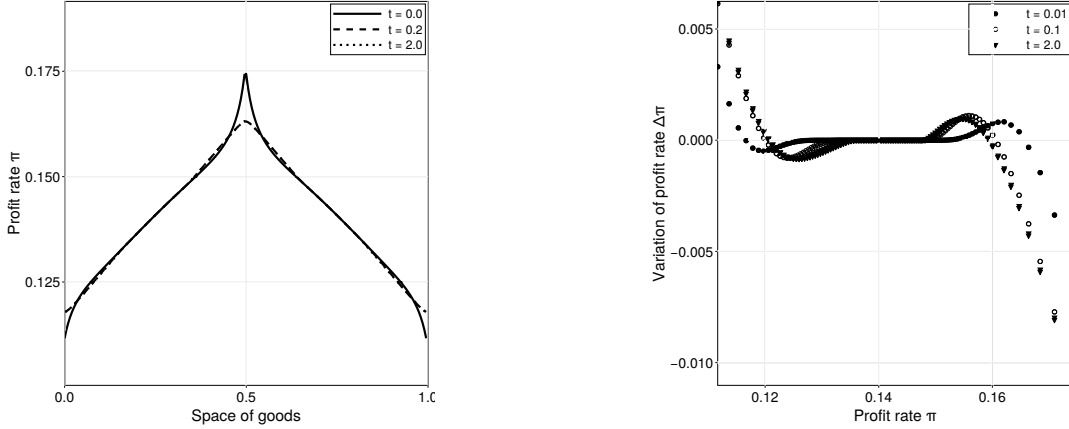
(B) The convergence of aggregate consumption for different values of the parameters β , σ , and η , relative to the baseline economy. The x -axis reports the time period, while the y -axis shows the logarithm of the ratio between consumption in the long-run competitive equilibrium and that observed in the economy at each period.

Figure 6b supports the validity of the gradient flow approach in characterizing the dynamics of aggregate consumption X . The x -axis reports the time period, while the y -axis shows the logarithm of the ratio between consumption in the long-run competitive equilibrium and that observed in the economy at each period, X^{EQ}/X . The downward trend observed for the baseline economy confirms that consumption increases over time, as expected, given that Condition (33) holds and $\eta > \beta - 1$ (see Equation (34)). In contrast, for the economy with $\eta = -0.5$, we have $\eta < \beta - 1$, and correspondingly, the ratio exhibits an upward trend indicating that, in this case, consumption tends to be minimized in the long-run competitive equilibrium.

²⁶ The L^2 distance is defined as the square root of the integral of the squared differences between the current distribution of firms and the long-run competitive equilibrium distribution.

4.2.3. *The economy with a fixed cost of reallocation.* Figure 7 illustrates the main effects of introducing a fixed cost of reallocation (we set $c_0 = 0.1$ in Equation (37), while keeping all other parameters at their baseline values). A comparison between Figures 5a and 7a reveals that the presence of fixed reallocation costs prevents profit rates from converging to a uniform distribution. In our numerical example, the long-run competitive distribution of profit rates remains markedly dispersed, ranging from 0.12 to 0.165. Figure 7b provides further insight into the dynamics of profit rates: even over an extended time horizon, a wide range of profit rates persists around the long-run competitive equilibrium value of 0.14. This reflects the existence of a plateau where reallocation does not occur, due to insufficient incentives for firms to move across sectors. This is evidenced by the flat slope of the profit rate distribution around 0.14 at the final period of the simulation ($t = 2$) in Figure 7a.

FIGURE 7. The economy with fixed costs of reallocation.



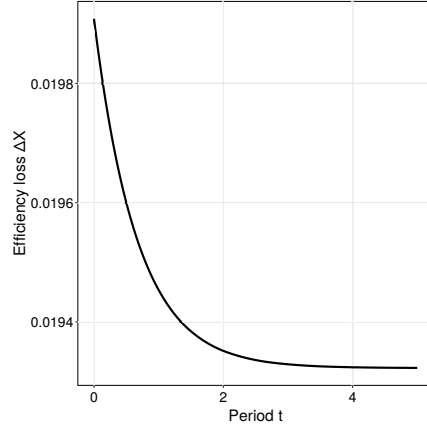
(A) The profit rate distribution at different periods in presence of a fixed cost of reallocation. On x -axis are reported the space of goods, while on y -axis the profit rates.

(B) The relationship between the sectoral profit rates at the time 0 (x -axis) and their change over time of the same (y -axis) in presence of fixed costs of reallocation.

4.2.4. *The cost of labour immobility.* We conclude our numerical experiments by quantifying the efficiency loss, proxied by a lower consumption, resulting from labour immobility (see Proposition 3.2). In Figure 8, this loss is greatest at $t = 0$, as the absence of instantaneous labour reallocation contributes maximally to the differential efficiency between mobile and immobile labour scenarios. The magnitude of this efficiency loss is approximately 2% relative to the long-run competitive equilibrium with mobile labour (given that $A(i) = 1 \forall i \in S$, we have $X^{EQ} = 1$, see Proposition 3.2). Although gains from firm reallocation are higher in

the less efficient (immobile labour) economy, leading to a gradual reduction in the efficiency gap, the loss remains positive and exceeds 1.92% even in the long run.

FIGURE 8. The loss in consumption due to the labour immobility



The dynamics of the difference in consumption between an economy with mobile labour and one with immobile labour. The x -axis represents the time period, while the y -axis shows the difference in consumption X between the two economies.

5. CONCLUSIONS

We have developed a robust framework for analysing the sectoral reallocation of firms (capital) in the presence of quadratic reallocation costs, using a gradient flow approach and the JKO scheme to characterize both short- and long-run competitive equilibria. Under mild assumptions, we prove convergence to the long-run competitive equilibrium and establish its efficiency properties. Our analysis highlights the critical role of the elasticity of substitution between goods and intra-sectoral externalities in shaping the distribution of firm sizes across sectors. Notably, we show that even modest fixed reallocation costs can give rise to multiple equilibria and path dependence on initial conditions. Labour mobility emerges as a key factor influencing efficiency in both the short and long run.

Overall, the model is sufficiently flexible to incorporate various extensions, including non-symmetric preferences over goods, intra-sectoral externalities, and labour immobility. A first empirical application to a large sample of EU firms offers supporting evidence for the relevance and applicability of our approach.

We are aware of the limitations of our model, many of which suggest promising avenues for future research. The assumption of myopic firms and, in particular, of static expectations, is one of the limitations, especially when analysing the properties of business cycles. [Sargent \(1993\)](#) suggests ways to address this, for example by introducing more sophisticated adaptive expectations on sectoral profits. Our conjecture is that such an extension may affect the properties of the dynamic path, but not the key features of the long-run competitive equilibrium.

Another natural extension involves jointly considering capital and labour mobility in the presence of frictions. This joint reallocation introduces non-trivial technical challenges but enables the exploration of richer and more complex dynamics. In particular, complementarities in production between these two factors may lead to the concentration of resources in a few sectors, where returns to individual inputs (wages and profit rates) are dynamically enhanced by continuous inflows of both capital and labour from other sectors ([Krugman, 1991, 1994](#)).

An extension would allow for positive saving and capital accumulation, raising the question of which sectors benefit from a growing capital stock. A key challenge in this context is defining the entry process for new firms and understanding how such dynamics shape sectoral reallocation ([Melitz, 2003](#)).

Related to this is the possibility of introducing a financial sector, which could facilitate capital mobility across sectors without incurring quadratic reallocation costs. In this setting, it would be important to revisit the nature of reallocation costs, potentially allowing for capital losses during intersectoral transfers ([Eisfeldt and Rampini, 2006](#)) and introducing an opportunity cost of investing in the production of goods. Finally, incorporating some form of nominal rigidity in the short-run adjustment of prices and sectoral wages could enhance the model's relevance for studying medium-run fiscal and monetary policy effects. Such rigidities may generate sluggish adjustment dynamics, making them particularly important for macroeconomic analysis ([Blanchard, 2025](#)).

In summary, our framework provides a solid foundation for analysing sectoral dynamics and offers a versatile tool for studying complex processes of factor reallocation. We are confident that it can serve as a basis for future extensions, further enhancing the applicability and explanatory power of our approach to understanding the dynamics of a capitalist economy.

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Some of the appendices (in particular Appendices B, D, E) can be included as online appendices.

APPENDIX A. PROOFS FOR THE BASELINE MODEL

Proof of Proposition 2.1 Observe first that, thanks to (6) we have that the output supply of each firm producing i is

$$(59) \quad q(i) = \beta^{\frac{\beta}{1-\beta}} A(i)^{\frac{1}{1-\beta}} p(i)^{\frac{\beta}{1-\beta}} w(i)^{-\frac{\beta}{1-\beta}}.$$

The total nominal income derived from the production of good i is therefore:

$$(60) \quad p(i)q(i)\mu(i) = \beta^{\frac{\beta}{1-\beta}} A(i)^{\frac{1}{1-\beta}} p(i)^{\frac{1}{1-\beta}} w(i)^{-\frac{\beta}{1-\beta}} \mu(i)$$

and the profits of each firm producing i

$$(61) \quad \pi(i) = \beta^{\frac{\beta}{1-\beta}} (1 - \beta) A(i)^{\frac{1}{1-\beta}} p(i)^{\frac{1}{1-\beta}} w(i)^{-\frac{\beta}{1-\beta}} = (1 - \beta) p(i)q(i),$$

i.e., a share of $1 - \beta$ of the firm's production goes to profits, while a share of β goes to wages (this is, of course, standard given the Cobb-Douglas form of the production function). The total profits of the sector i are therefore equal to:

$$(62) \quad \pi(i)\mu(i) = \mu(i)\beta^{\frac{\beta}{1-\beta}} (1 - \beta) A(i)^{\frac{1}{1-\beta}} p(i)^{\frac{1}{1-\beta}} w(i)^{-\frac{\beta}{1-\beta}}.$$

To prove (12) we have just to observe that, since the share of wages is constant and equal to β across the economy, the total mass of nominal wages in the economy is βY and thus:

$$(63) \quad w = \beta \frac{Y}{L} = \beta Y,$$

where the last equality follows from the assumption $L = 1$. Since we normalized $Y = 1$ we have the claim.

To prove (11) we can first observe that market clearing condition in good market i gives, using (2) and then (59) i.e.

$$(64) \quad \frac{Y}{P} \left(\frac{p(i)}{P} \right)^{-\sigma} = q(i)\mu(i) = \beta^{\frac{\beta}{1-\beta}} A(i)^{\frac{1}{1-\beta}} p(i)^{\frac{\beta}{1-\beta}} w(i)^{-\frac{\beta}{1-\beta}} \mu(i),$$

from which the equilibrium price for good i (using that $Y = 1$):

$$(65) \quad p(i) = \left(P^{1-\sigma} A(i)^{\frac{1}{1-\beta}} \mu(i) \right)^{-\frac{1-\beta}{\sigma(1-\beta)+\beta}}$$

Thus, we can take equation (65), raise it to the power of $1 - \sigma$, and integrate and, using the definition of P given in (3), we have:

$$P^{1-\sigma} = \left(\int_0^n p(i)^{1-\sigma} di \right) = P^{-\frac{(1-\sigma)^2(1-\beta)}{\sigma(1-\beta)+\beta}} \left(\int_0^n \left(A(i)^{\frac{1}{1-\beta}} \mu(i) \right)^{1-\frac{1}{\sigma(1-\beta)+\beta}} di \right).$$

We then find

$$P = \left(\int_0^n \left(A(i)^{\frac{1}{1-\beta}} \mu(i) \right)^{1-\frac{1}{\sigma(1-\beta)+\beta}} di \right)^{\frac{\sigma(1-\beta)+\beta}{1-\sigma}}.$$

that is (11). So, again from (65),

$$\begin{aligned} (66) \quad p(i) &= P^{1-\frac{1}{\sigma(1-\beta)+\beta}} \left(A(i)^{\frac{1}{1-\beta}} \mu(i) \right)^{-\frac{1-\beta}{\sigma(1-\beta)+\beta}} \\ &= \left(\int_0^n \left(A(j)^{\frac{1}{1-\beta}} \mu(j) \right)^{1-\frac{1}{\sigma(1-\beta)+\beta}} dj \right)^{-(1-\beta)} \left(A(i)^{\frac{1}{1-\beta}} \mu(i) \right)^{-\frac{1-\beta}{\sigma(1-\beta)+\beta}}. \end{aligned}$$

that is (9). So, from (61)

$$\begin{aligned} (67) \quad \pi(i) &= (1-\beta) \beta^{\frac{\beta}{1-\beta}} A(i)^{\frac{1}{1-\beta}} p(i)^{\frac{1}{1-\beta}} \beta^{\frac{-\beta}{1-\beta}} = (1-\beta) A(i)^{\frac{1}{1-\beta}} p(i)^{\frac{1}{1-\beta}} \\ &= \frac{(1-\beta) A(i)^{\frac{1}{1-\beta}} \left(1-\frac{1}{\sigma+\beta-\sigma\beta} \right) \mu(i)^{\frac{-1}{\sigma(1-\beta)+\beta}}}{\left(\int_0^n \left(A(j)^{\frac{1}{1-\beta}} \mu(j) \right)^{1-\frac{1}{\sigma+\beta-\sigma\beta}} dj \right)} \end{aligned}$$

that is (10) and,

$$\Pi = \int_0^n \pi(i) \mu(i) di = \frac{\int_0^n (1-\beta) A(i)^{\frac{1}{1-\beta}} \left(1-\frac{1}{\sigma+\beta-\sigma\beta} \right) \mu(i)^{\frac{-1}{\sigma(1-\beta)+\beta}} di}{\left(\int_0^n \left(A(j)^{\frac{1}{1-\beta}} \mu(j) \right)^{1-\frac{1}{\sigma+\beta-\sigma\beta}} dj \right)} = (1-\beta).$$

Finally, the total production in each sector is

$$\mu(i)q(i) = A(i)^{\frac{1}{1-\beta}} p(i)^{\frac{\beta}{1-\beta}} \mu(i) = A(i)^{\frac{1}{1-\beta}} \left(\int_0^n \left(A(j)^{\frac{1}{1-\beta}} \mu(j) \right)^{1-\frac{1}{\sigma+\beta-\sigma\beta}} dj \right)^{-\beta} \left(A(i)^{\frac{1}{1-\beta}} \mu(i) \right)^{\frac{-\beta}{\sigma(1-\beta)+\beta}} \mu(i),$$

while the revenues are

$$\mu(i)q(i)p(i) = A(i)^{\frac{1}{1-\beta}} \left(\int_0^n \left(A(j)^{\frac{1}{1-\beta}} \mu(j) \right)^{1-\frac{1}{\sigma+\beta-\sigma\beta}} dj \right)^{-1} \left(A(i)^{\frac{1}{1-\beta}} \mu(i) \right)^{\frac{-1}{\sigma(1-\beta)+\beta}} \mu(i).$$

and

$$L(i) = \mu(i)l(i) = \mu(i)A(i)^{\frac{1}{1-\beta}} \left(\int_0^n \left(A(j)^{\frac{1}{1-\beta}} \mu(j) \right)^{1-\frac{1}{\sigma+\beta-\sigma\beta}} dj \right)^{-1} \left(A(i)^{\frac{1}{1-\beta}} \mu(i) \right)^{\frac{-1}{\sigma(1-\beta)+\beta}}$$

We have then proved all the statements of the proposition. $\square$

Proof of Theorem 2.2. In this proof, to lighten the notation, we will define $\rho \equiv \sigma(1-\beta) + \beta$. Notice that $\rho \in (\sigma, 1)$ or $\rho \in (1, \sigma)$ depending on whether $\sigma \leq 1$.

We start by looking at the following PDE:

$$(68) \quad \begin{cases} \partial_t V(t, i) = -(1 - \beta) \partial_i \left(V(t, i) \partial_i \left(A(i)^{\frac{1}{1-\beta}} (1 - \frac{1}{\rho}) V(t, i)^{-1/\rho} \right) \right) \\ V(t, 0) = V(t, n) \\ V(0, i) = \mu_0(i). \end{cases}$$

We develop the right-hand side of the (68) in more explicit terms for later reference. If we avoid writing the dependences (t, i) and we denote $z = \frac{1}{1-\beta} \left(1 - \frac{1}{\rho} \right)$ We have that

$$(69) \quad \begin{aligned} & -(1 - \beta) \partial_i \left(V(t, i) \partial_i \left(A(i)^{\frac{1}{1-\beta}} (1 - \frac{1}{\rho}) V(t, i)^{-1/\rho} \right) \right) \\ &= -(1 - \beta) V^{-1/\rho} \left[(1 - 2/\rho) (z A^{z-1}) (\partial_i A) (\partial_i V) + \frac{1}{\rho^2} A^z (\partial_i V)^2 \right. \\ & \quad \left. + z(z - 1) A^{z-2} (\partial_i A)^2 V + z A^{z-1} (\partial_i^2 A) V - \frac{1}{\rho} A^z (\partial_i^2 V) \right] =: f(V, \partial_i V, \partial_i^2 V). \end{aligned}$$

Thanks to Theorem 1.2 of [Iacobelli et al. \(2019\)](#) (the results hold for Neumann and also periodic boundary conditions so they include our case) it has a unique solution in $L^2_{loc}([0, +\infty), H^1(S))$ and thanks to their Theorem 1.3 there exists $M > 0$ such that, for all $t \geq 0$

$$(70) \quad \frac{1}{M} < V(t, i) < M, \quad \text{for all } i \in S.$$

This fact guarantees further regularity of V :

- (i) The positivity result (70) ensure that the operator f defined in (69) is uniformly elliptic and then one can use the standard results for parabolic equations for recursively get (using for instance Theorem 15.2 page 81 and then 6.1 page 33 of [Lions and Magenes, 1972](#) or Krylov-Safonov and then Schauder estimates) the regularity in the interior and that $V \in C([0, +\infty), C^2(S))$.
- (ii) For positive and regular initial data as the μ_0 considered in the text of the theorem one can gain some additional regularity at $t = 0$. In particular, we can for instance apply Theorem 8.1.3 of [Lunardi \(1995\)](#)²⁷ to be sure that there exists α such that, for all²⁸ $T > 0$,

$$(71) \quad V(t) \in C^{1+\alpha}([0, T], C(S)).$$

²⁷ We use Theorem 8.1.3 of [Lunardi \(1995\)](#) specifying the abstract setting as follows: $X = C(S)$, $D = C^2(S)$, $\mathcal{O} = \{v \in D : \frac{1}{M} < v(i) < M, \text{ for all } i \in S\}$, $F(t, v)(\xi) = f(v(\xi), v'(\xi), v''(\xi))$ where f is defined in (69). Remark that, since S is compact, the condition $A \in C^2(S)$ implies in particular that the first and the second derivatives of A are bounded. As remarked below equation (8.0.3) page 287 in [Lunardi \(1995\)](#) in our specification, the condition (8.0.3) is verified as long as $\frac{\partial f}{\partial q}(u, p, q) > 0$ which is assured by the positivity of A and of V . The Lipschitz and the Holder conditions (8.1.1) page 290 are verified thanks to the bounds in the definition of $\mathcal{O}$, the continuity and the Fréchet differentiability are assured near μ_0 (the regularity result is local in time) thanks to its regularity and its boundedness. Again the regularity of μ_0 ensures that the condition (8.1.18), page 295 is verified.

²⁸ The result of Theorem 8.1.3 of [Lunardi \(1995\)](#) only ensures this fact until to a small $\delta > 0$ but, since all the boundedness are homogeneous in time, the result extends to $[0, T]$.

We want to use V to obtain a solution of (21). If μ (we will prove its existence at the end) is a $C^{1,2}$ solution of (21) and there exist two regular functions $\varphi: \mathbb{R}^+ \rightarrow \mathbb{R}^+$ and $V: \mathbb{R}^+ \times S \rightarrow \mathbb{R}$ such that $\mu(t, i) = V(\varphi(t), i)$, then μ satisfies

$$\partial_t \mu(t, i) = \varphi'(t) \partial_t V(\varphi(t), i).$$

Moreover, if $\varphi: \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is monotonously increasing, $\varphi(0) = 0$ and

$$(72) \quad \varphi'(t) = \frac{1}{\left(\int_0^n [A(j)^{1/(1-\beta)} \mu(t, j)]^{1-\frac{1}{\rho}} dj \right)} = \frac{1}{\left(\int_0^n [A(j)^{1/(1-\beta)} V(\varphi(t), j)]^{1-\frac{1}{\rho}} dj \right)},$$

then $V(\varphi(t), i)$ is a solution to Equation (21). Hence, to prove existence and uniqueness to (21) it is enough, starting from that of (68), to show the existence and uniqueness of the time change $\varphi(t)$ i.e. the existence of a unique φ such that

$$\begin{cases} \varphi'(t) = G(\varphi(t)) := \frac{1}{\int_0^n [A(j)^{1/(1-\beta)} V(\varphi(t), j)]^{1-\frac{1}{\rho}} dj} \\ \varphi(0) = 0 \end{cases}$$

To prove that this equation has a unique solution, it is enough to prove that G is Lipschitz-continuous. So we need to estimate

$$\begin{aligned} |G(x) - G(y)| &= \left| \frac{1}{\int_0^n [A(j)^{1/(1-\beta)} V(x, j)]^{1-\frac{1}{\rho}} dj} - \frac{1}{\int_0^n [A(j)^{1/(1-\beta)} V(y, j)]^{1-\frac{1}{\rho}} dj} \right| \\ &= \left| \frac{\int_0^n [A(j)^{1/(1-\beta)} V(x, j)]^{1-\frac{1}{\rho}} dj - \int_0^n [A(j)^{1/(1-\beta)} V(y, j)]^{1-\frac{1}{\rho}} dj}{\left(\int_0^n [A(j)^{1/(1-\beta)} V(x, j)]^{1-\frac{1}{\rho}} dj \right) \left(\int_0^n [A(j)^{1/(1-\beta)} V(y, j)]^{1-\frac{1}{\rho}} dj \right)} \right|. \end{aligned}$$

The denominator of this expression, thanks to (70) and the strict positivity and boundedness of A , is always greater than some constant $1/C > 0$ so the previous expression is smaller than

$$\begin{aligned} C \cdot \left| \int_0^n [A(j)^{1/(1-\beta)} V(x, j)]^{1-\frac{1}{\rho}} dj - \int_0^n [A(j)^{1/(1-\beta)} V(y, j)]^{1-\frac{1}{\rho}} dj \right| &= \\ &= \left| \int_0^n [A(j)^{1/(1-\beta)} V(x, j)]^{1-\frac{1}{\rho}} - [A(j)^{1/(1-\beta)} V(y, j)]^{1-\frac{1}{\rho}} dj \right|. \end{aligned}$$

Again, thanks to (70) and to the strict positivity and boundedness of A , we know that the quantity $[A(j)^{1/(1-\beta)} V(x, j)]$ stays uniformly away from zero. Since the map $z \mapsto z^{1-\frac{1}{\rho}}$ is Lipschitz-continuous on bounded subsets of $(0, +\infty)$, it follows that applying this transformation yields a Lipschitz functional. Therefore, possibly adjusting the constant C (also depending on the bounds on A), the previous expression is smaller than

$$(73) \quad \leq C \int_0^n |V(x, j) - V(y, j)| dj.$$

Since (71) holds we know that

$$\sup_{i \in S} \sup_{t \in [x, y]} \left| \frac{\partial V}{\partial t} \right| (t, i) : N < +\infty$$

So (73) is smaller than

$$C \int_0^n N |x - y| \, dj = CNn |x - y|$$

and then G is Lipschitz continuous on all intervals $[0, T]$. Therefore, the time change exists, and it is unique and so is the solution μ of (21). $\square$

Proof of Theorem 2.3. Introduce $\rho \equiv \sigma(1 - \beta) + \beta$. Consider the equation

$$\begin{cases} \partial_t \nu(t, i) = -\frac{(1 - \rho)}{\rho} \partial_i \left(\nu(t, i) \partial_i \left(A(i)^{\frac{1}{1-\beta}} (1 - \frac{1}{\rho}) \nu(i)^{-1/\rho} \right) \right) \\ \nu(0, t) = \nu(n, t) \\ \nu(i, 0) = \nu_0(i). \end{cases}$$

The results of Iacobelli et al. (2019) (see in particular page 1171 and subsequent²⁹) ensure that its (unique, weak, see Theorem 1.2 page 1170) solution can be characterized as the gradient flow (defined as the limit of the Jordan-Kinderlehrer-Otto scheme, see Lemma 2.2 page 1174 of Iacobelli et al. (2019)) of the functional

$$(74) \quad \mathcal{G}(\nu) := \int_0^n A(i)^{\frac{1}{1-\beta}} (1 - \frac{1}{\rho}) \nu(i)^{\frac{\rho-1}{\rho}} \, di$$

i.e.

$$\frac{d\nu}{dt} = \nabla_{W_2} \mathcal{G}(\nu(t)).$$

If we consider

$$\mathcal{F}(\nu) = \frac{\rho}{1 - \rho} (1 - \beta) \ln(\mathcal{G}(\nu))$$

we have

$$\nabla_{W_2} \mathcal{F}(\mu(t)) = \frac{\rho}{1 - \rho} (1 - \beta) \frac{\nabla_{W_2} \mathcal{G}(\mu(t))}{\mathcal{G}(\mu(t))}.$$

which corresponds to the expression appearing in the right hand side of (21) so that

$$\frac{d\mu}{dt} = \nabla_{W_2} \mathcal{F}(\nu(t)).$$

and this concludes the proof. $\square$

Proof of Corollary 2.4. The fact that the gradient flow evolves along the curves of maximal slope is a very general result; see, for example, Theorem 11.1.3, page 283 of Ambrosio et al. (2005). Therefore, in particular,

²⁹ To adapt our notation to theirs (see (1.1) in Iacobelli et al. (2019)) we can set

$$\rho(x) = A(x)^{\frac{1}{1-\beta} 1 - \frac{1}{\rho}}, \quad \text{and } r = \frac{1}{\rho} - 1.$$

$t \mapsto \mathcal{F}(\mu(t))$ is decreasing. Moreover, we have

$$P(t) = \left(\int_0^n \left[A(i)^{1/(1-\beta)} \mu(t, i) \right]^{1-1/\rho} di \right)^{\frac{\rho}{1-\rho}} = \exp \left(\frac{1-\rho}{\rho} \frac{1}{1-\beta} \mathcal{F}(\mu(t)) \right)^{\frac{\rho}{1-\rho}}.$$

Thus, $t \mapsto P(t)$ decreases along the trajectories. The behaviours of $X(t)$ and $U(t)$ follow from this fact and from the relations (14) and (15). $\square$

Proof of Theorem 2.5. Finding a long-run competitive equilibrium amounts to identifying a stationary state of the dynamics in (21). Indeed, any long-run competitive equilibrium must necessarily also be a short-run competitive equilibrium, otherwise profit and utility maximization would be violated or markets would fail to clear—so that profits are given by (10). If, as required in the definition of a long-run competitive equilibrium, no firm has an incentive to switch sectors, i.e. it chooses $v = 0$ in Problem (17), then π must be constant in i . This means that there exists some positive real number C_∞ such that

$$\mu(i)^{-\frac{1}{\rho}} A(i)^{\frac{1}{1-\beta} \frac{\rho-1}{\rho}} = C_\infty,$$

and, using the fact that μ integrates to one, (27) distribution can be derived.

On the other hand, any stationary state μ of (21) must satisfy that the term

$$\Psi(i) := (1-\beta) \frac{A(i)^{\frac{1}{1-\beta} \frac{\rho-1}{\rho}} \mu(i)^{-\frac{1}{\rho}}}{\int_0^n \left[A(j)^{\frac{1}{1-\beta}} \mu(j) \right]^{\frac{\rho-1}{\rho}} dj}$$

is constant on S . Indeed, we can write (21) as a continuity equation:

$$\partial_t \mu = -\partial_i J, \quad J(i) := \mu(i) \partial_i \Psi(i).$$

At stationarity $\partial_t \mu = 0$, hence $\partial_i J = 0$ and $J(i) \equiv c$ for some constant c . Integrating over S we have

$$0 = \int_S \partial_i \Psi di = \int_S \frac{J}{\mu} di = c \int_S \frac{1}{\mu} di,$$

(the first equality holds since Ψ is periodic) and since $\mu > 0$ the integral is strictly positive, forcing $c = 0$. Hence $J \equiv 0$, so with $\mu > 0$ we get $\partial_i \Psi = 0$ everywhere, i.e. $\Psi(i) \equiv C_\infty$. Equivalently,

$$\mu(i)^{-\frac{1}{\rho}} A(i)^{\frac{1}{1-\beta} \frac{\rho-1}{\rho}} = C_\infty.$$

This is exactly the condition of no incentive to reallocate for firms and, again using the fact that μ integrates to one, we get (27). Once we couple this condition with the prices and the quantity found in Proposition 2.1 in (2) and (6) we get we have, by the short-run competitive equilibrium characterization, that all markets clear, consumers and firms are maximizing their utility and workers have no incentive to reallocate. So we have a long-run competitive equilibrium.

To show the convergence part of the Theorem, first of all, consider the time change $\varphi(t)$ introduced in the proof of Theorem 2.2, and defined in (72). Notice that, by equation (70), such a φ will automatically be strictly increasing and with bounded first derivative, i.e.

$$c_\varphi \leq \varphi'(t) \leq C_\varphi$$

for some constants $c_\varphi > 0, C_\varphi > 0$, so that $\varphi \in C(\mathbb{R}^+)$ and $\lim_{t \rightarrow \infty} \varphi(t) = +\infty$. This property is fundamental since it guarantees the uniqueness of the solution of (21) from that of $\varphi(t)$. Moreover, since $\varphi(0) = 0$ it also implies that $\varphi(t) \geq K_\varphi \cdot t$ for some $K_\varphi > 0$.

Consider Equation (68). Thanks to Theorem 1.4 of Iacobelli et al. (2019) for the function V , solution to (68) it holds³⁰

$$(75) \quad \|V(t, \cdot) - \mu^{EQ}(\cdot)\|_{L^2(S)} \leq C_1 e^{-C_2 t},$$

(which also implies the uniqueness of the steady state). Therefore, since $\mu(t, i) = V(\varphi(t), i)$ one immediately has

$$(76) \quad \|\mu(t, \cdot) - \mu^{EQ}(\cdot)\|_{L^2(S)} \leq C_1 e^{-C_2 \varphi(t)}.$$

However, this immediately translates to

$$(77) \quad \|\mu(t, \cdot) - \mu^{EQ}(\cdot)\|_{L^2(S)} \leq C_1 e^{-C_2 K_\varphi t}$$

which ends the proof.

To derive the expressions for $\pi(i)$, $Y(i)$, and $L(i)$, it suffices to substitute the expression from (27) into the corresponding formulas given in Proposition 2.1. $\square$

Proof of Theorem 2.6. We define $\rho := \sigma(1 - \beta) + \beta$ and notice that $\frac{\rho-1}{\sigma-1} > 0$ for all $\sigma > 0$ with $\sigma \neq 1$.

Consider for the case $\sigma > 1$, i.e. $\rho > 1$ (the case $0 < \sigma < 1$ follows from similar arguments) and consider the quantity

$$X(\mu) = \left(\int_0^n \left[A(i)^{\frac{1}{1-\beta}} \mu(i) \right]^{\frac{\rho-1}{\rho}} di \right)^{\frac{\rho}{\sigma-1}} = \left(\int_0^n \left[A(i)^{\frac{1}{1-\beta}} \mu(i) \right]^{\frac{\rho-1}{\rho}} di \right)^{\frac{\rho}{\rho-1} \frac{\rho-1}{\sigma-1}}.$$

Since the exponent outside of the integral X is maximized if and only if the value of the integral is maximized, so consider μ , such that $\int \mu = 1$. We define $\bar{A}(i) := A(i)^{\frac{1}{1-\beta} \frac{\rho-1}{\rho}}$ and $\gamma = \frac{\rho-1}{\rho}$. Since $0 < \gamma < 1$ the γ -power function is concave and then one

$$\int_0^n \bar{A}(i) \mu(i)^\gamma - \bar{A}(i) \mu^{EQ}(i)^\gamma di \leq \int_0^n \bar{A}(i) \gamma \mu^{EQ}(i)^{\gamma-1} [\mu(i) - \mu^{EQ}(i)] di.$$

³⁰ Notice that, the steady state of (68) is the same of (21).

On the other hand

$$(78) \quad \int_0^n \bar{A}(i) \gamma \mu^{EQ}(i)^{\gamma-1} [\mu(i) - \mu^{EQ}(i)] di = \frac{\int_0^n \bar{A}(i) \gamma A(i)^{\frac{\rho-1}{1-\beta}(\gamma-1)} [\mu(i) - \mu^{EQ}(i)] di}{\left(\int_0^n A(j)^{\frac{\sigma(1-\beta)+\beta-1}{1-\beta}} dj \right)^{\gamma-1}}.$$

and

$$A(i)^{\frac{\rho-1}{1-\beta}(\gamma-1)} = A(i)^{\frac{\rho-1}{1-\beta} \frac{-1}{\rho}} = \bar{A}(i)^{-1}$$

so that (78) becomes

$$(79) \quad = \frac{\int_0^n \gamma [\mu(i) - \mu^{EQ}(i)] di}{\left(\int_0^n A(j)^{\frac{\sigma(1-\beta)+\beta-1}{1-\beta}} dj \right)^{\gamma-1}} = 0.$$

Therefore for each μ such that $\int \mu = 1$ one has

$$(80) \quad \int_0^n \bar{A}(i) \mu(i)^\gamma \leq \bar{A}(i) \mu^{EQ}(i)^\gamma di$$

which ends the proof.

A similar argument holds for $0 < \sigma < 1$. In this case, $\gamma = (\rho - 1)/\rho < 0$ and the outer exponent $\beta = \rho/(\sigma - 1) < 0$. Thus, $x \mapsto x^\gamma$ is convex, and maximizing $X(\mu) = (\int \bar{A}\mu^\gamma)^\beta$ requires minimizing the integral. The convexity inequality, combined with the previously shown zero value of the first-order term, confirms that μ^{EQ} minimizes the integral, thus maximizing $X(\mu)$. $\square$

APPENDIX B. PROPOSITIONS AND PROOFS FOR THE EXTENSIONS OF THE BASELINE MODEL PRESENTED IN SUBSECTION 2.5 (TOGETHER WITH A NON-SYMMETRIC PREFERENCE EXTENSION)

B.1. The economy with intra-sectoral externalities. This section contains Proposition B.1 which collects the properties of the short-run competitive equilibrium in presence of intra-sectoral externalities. The equilibrium is similar to the case without externalities, except that $A(i)$ is replaced by $A_0(i)\mu(i)^\eta$.

PROPOSITION B.1 (The short-run competitive equilibrium with externalities) *In the short-run competitive equilibrium with intra-sectoral externalities:*

$$w = \beta,$$

$$(81) \quad p(i) = \frac{[A_0(i)\mu(i)^{1-\beta+\eta}]^{-\frac{1}{\sigma(1-\beta)+\beta}}}{\left(\int_0^n [A_0(j)\mu(j)^{1-\beta+\eta}]^{\frac{1}{1-\beta}(1-\frac{1}{\sigma(1-\beta)+\beta})} dj \right)^{(1-\beta)}},$$

$$(82) \quad P = \left(\int_0^n [A_0(i)\mu(i)^{1-\beta+\eta}]^{\frac{\sigma-1}{\sigma(1-\beta)+\beta}} di \right)^{\frac{\sigma(1-\beta)+\beta}{1-\sigma}},$$

$$(83) \quad \pi(i) = (1 - \beta) \frac{A_0(i)^{\frac{\sigma-1}{\sigma(1-\beta)+\beta}} \mu(i)^{\frac{\eta(\sigma-1)-1}{\sigma(1-\beta)+\beta}}}{\int_0^n [A_0(j)\mu(j)^{1-\beta+\eta}]^{\frac{\sigma-1}{\sigma(1-\beta)+\beta}} dj}$$

the production in sector i is

$$(84) \quad Q(i) = \frac{[A_0(i)\mu(i)^{1-\beta+\eta}]^{\frac{\sigma}{\sigma(1-\beta)+\beta}}}{\left(\int_0^n [A_0(j)\mu(j)^{1-\beta+\eta}]^{\frac{\sigma-1}{\sigma(1-\beta)+\beta}} dj\right)^\beta},$$

while the value of production and employment

$$(85) \quad Y(i) \equiv L(i) = \frac{[A_0(i)\mu(i)^{1-\beta+\eta}]^{\frac{\sigma-1}{\sigma(1-\beta)+\beta}}}{\int_0^n [A_0(j)\mu(j)^{1-\beta+\eta}]^{\frac{\sigma-1}{\sigma(1-\beta)+\beta}} dj}.$$

Proof. The proof follows directly from Proposition 2.1 by using that $A(i) = A_0(i)\mu(i)^\eta$. $\square$

As before $\Pi = (1 - \beta)$.

B.1.1. *A fixed cost of reallocation.*

Proof that the distribution μ described in (39) is an equilibrium when the condition (40) is verified. As already mentioned, to ensure that the distribution has a total mass equal to one, we need to set

$$C_\alpha = \frac{1 - \alpha}{2^\alpha}.$$

We also compute

$$(86) \quad \partial_x \mu(x) = \begin{cases} -\frac{\alpha(1-\alpha)}{2^\alpha} \frac{1}{x^{\alpha+1}} & \text{for } x \in (0, 1/2), \\ -\frac{\alpha(1-\alpha)}{2^\alpha} \frac{1}{(1-x)^{\alpha+1}} & \text{for } x \in (1/2, 1), \end{cases}$$

The expression for profits, defining $\rho = \sigma(1 - \beta) + \beta$, is given by

$$\begin{aligned} \pi(x) &= (1 - \beta) \left(\int_0^n [A(i)^{1/(1-\beta)} \mu(i)]^{1 - \frac{1}{\sigma+\beta-\sigma\beta}} di \right) A(i)^{\frac{1}{1-\beta} (1 - \frac{1}{\sigma+\beta-\sigma\beta})} \mu(i)^{-1/(\sigma(1-\beta)+\beta)} \\ &= (1 - \beta) \int_0^1 \mu(x)^{\frac{\rho-1}{\rho}} dx \mu(x)^{-\frac{1}{\rho}}, \end{aligned}$$

which leads to

$$(87) \quad \partial_x \pi(x) = (1 - \beta) \int_0^1 \mu(x)^{\frac{\rho-1}{\rho}} dx \left(-\frac{1}{\rho} \mu(x)^{-\frac{1}{\rho}-1} \right) \partial_x \mu(x).$$

By symmetry,

$$\begin{aligned} \int_0^1 \mu(x)^{\frac{\rho-1}{\rho}} dx &= 2 \int_0^{1/2} \mu(x)^{\frac{\rho-1}{\rho}} dx = 2 \left(\frac{1-\alpha}{2^\alpha} \right)^{\frac{\rho-1}{\rho}} \int_0^{1/2} \frac{1}{x^{\alpha(\frac{\rho-1}{\rho})}} dx \\ &= 2 \left(\frac{1-\alpha}{2^\alpha} \right)^{\frac{\rho-1}{\rho}} \frac{2^{\alpha(\frac{\rho-1}{\rho})-1}}{1-\alpha(\frac{\rho-1}{\rho})} = \frac{\rho(1-\alpha)^{\frac{\rho-1}{\rho}}}{\alpha(1-\rho)+\rho}. \end{aligned}$$

The expression for the derivative of profits (87) then becomes

$$\begin{aligned} \partial_x \pi(x) &= (1-\beta) \underbrace{\frac{\rho(1-\alpha)^{\frac{\rho-1}{\rho}}}{\alpha(1-\rho)+\rho}}_{\text{integral}} \underbrace{\left(-\frac{1}{\rho} \right) \left(\frac{1-\alpha}{2^\alpha} \right)^{-\frac{1}{\rho}-1} \frac{1}{x^{\alpha(-\frac{1}{\rho}-1)}}}_{\mu \text{ exponent}} \underbrace{(-\alpha) \left(\frac{1-\alpha}{2^\alpha} \right) \frac{1}{x^{\alpha+1}}}_{\partial_x \mu(x)} = \\ &= (1-\beta) \frac{(1-\alpha)^{\frac{\rho-1}{\rho}}}{\alpha(1-\rho)+\rho} \left(\frac{1-\alpha}{2^\alpha} \right)^{-\frac{1}{\rho}} \alpha \frac{1}{x^{1-\frac{\alpha}{\rho}}} = \frac{\alpha(1-\beta)}{2^\alpha} \frac{(1-\alpha)^{\frac{\rho-2}{\rho}}}{\alpha(1-\rho)+\rho} x^{\frac{\alpha}{\rho}-1}. \end{aligned}$$

If we take $\rho \in (0, 1)$ (i.e., $\sigma < 1$) and $\alpha > \rho$, then $\frac{\alpha}{\rho} - 1 > 0$, and the maximum of $\partial_x \pi(x)$ occurs at $x = 1/2$, with value

$$\frac{\alpha(1-\beta)}{2^{\alpha+\frac{\alpha}{\rho}-1}} \frac{(1-\alpha)^{\frac{\rho-2}{\rho}}}{\alpha(1-\rho)+\rho} = \frac{\alpha(1-\beta)}{2^{\alpha+\frac{\alpha}{\rho}-1}} \frac{1}{\alpha(1-\rho)+\rho} \frac{1}{(1-\alpha)^{\frac{2-\rho}{\rho}}}.$$

Therefore, when $0 < \rho < \alpha < 1$, if

$$\frac{\alpha(1-\beta)(1-\alpha)^{1-\frac{2}{\rho}}}{2^{\alpha+\frac{\alpha}{\rho}-1}[\alpha(1-\rho)+\rho]} \leq \sqrt{2c_0},$$

the distribution (39) is an equilibrium distribution for the model with the fixed cost of reallocation. $\square$

B.1.2. Non-symmetric preferences. In this section, we examine what happens when preferences are not symmetric over the range of goods $[0, n]$. Specifically, assume that the preferences take the form:

$$(88) \quad U = u(X), \quad \text{with} \quad X := \left(\frac{1}{n} \int_0^n \gamma(i) x(i)^{\frac{\sigma-1}{\sigma}} di \right)^{\frac{\sigma}{\sigma-1}},$$

for some strictly positive (and continuous and bounded) function $\gamma: S \rightarrow \mathbb{R}$, where $\gamma(i) > 0$, $\gamma \in C^2(S)$ and $\int_0^n \gamma(j) dj = n$. $\gamma(i)$ represents an index of the consumer's specific preference for good i , whose magnitude positively affects its demand and, consequently, the share of total expenditure allocated to the purchase of that good. Appendix B.1.3 shows that the consumer's problem can be transformed into the same of Proposition 2.1 by an appropriate rescaling of quantity of consumed good $x(i)$ and technology $A(i)$ (see Proposition B.2 in Appendix B.1.3 for details).

The key distinguishing feature of this extension is the positive relationship between the share of income spent on purchasing good i $Y(i)$ and $\gamma(i)$, i.e.:

$$(89) \quad \frac{p(i)x(i)}{y} = Y(i) = \gamma(i)^\sigma \left[\frac{p(i)}{P} \right]^{1-\sigma},$$

which reflects in a positive relationship also between profits and $\gamma(i)$:

$$(90) \quad \pi(i) = (1 - \beta) \left\{ \frac{\gamma(i)^{\frac{\sigma}{\sigma(1-\beta)+\beta}} A(i)^{\frac{\sigma(\sigma-1)}{\sigma(1-\beta)+\beta}} \mu(i)^{-\frac{1}{\sigma(1-\beta)+\beta}}}{\int_0^n [\gamma(j)^{\frac{\sigma}{\sigma-1}} A(j) \mu(j)^{1-\beta}]^{\frac{\sigma-1}{\sigma(1-\beta)+\beta}} dj} \right\},$$

The resulting long-run competitive equilibrium distribution of $\mu^{EQ}(i)$, $Y(i)^{EQ}$ and $L(i)^{EQ}$, i.e.

$$(91) \quad \mu^{EQ}(i) = Y(i)^{EQ} = L(i)^{EQ} = \frac{\gamma(i)^\sigma A(i)^{\sigma-1}}{\int_0^n \gamma(j)^\sigma A(j)^{\sigma-1} dj},$$

highlights that also preferences contributes to determine the relative size of different sectors but still remains that labour productivity should converge to one in each sector.

B.1.3. Proofs for the non-symmetric preferences. If we rescale each variety by its taste weight

$$\tilde{x}(i) = x(i) \gamma(i)^{\frac{\sigma}{\sigma-1}}, \quad \tilde{A}(i) = A(i) \gamma(i)^{-\frac{\sigma}{\sigma-1}},$$

then the CES aggregator becomes identical to the symmetric case treated in Proposition [2.1](#)

$$C = \left(\frac{1}{n} \int_0^n \tilde{x}(i)^{\frac{\sigma-1}{\sigma}} di \right)^{\frac{\sigma}{\sigma-1}}.$$

Production expressed in the new unit of measure is simply

$$(92) \quad \tilde{q}(i) = \tilde{A}(i) l(i)^\beta,$$

so the entire equilibrium of Proposition [2.1](#) applies *verbatim* after the single substitution

$$A(i) \longrightarrow \tilde{A}(i).$$

When the solution is translated back to the original units one finds

$$p(i) = \tilde{p}(i) \gamma(i)^{\frac{\sigma}{\sigma-1}}, \quad Q(i) = \tilde{Q}(i) \gamma(i)^{-\frac{\sigma}{\sigma-1}},$$

so that the price re-introduces the factor $\gamma(i)$ while the physical quantity is rescaled in the opposite direction. Because these two factors cancel in $Y(i) = p(i)Q(i)$, all other expressions (revenue, profits, the wage and welfare) remain exactly those obtained under the substitution. This observation shows gives, starting from the results of Proposition [2.1](#), the following result (a complete proof is provided)

PROPOSITION B.2 (The short-run competitive equilibrium with non-symmetric preferences) *In the short-run competitive equilibrium of economy with non-symmetric preferences:*

$$w = \beta,$$

$$\begin{aligned}
(93) \quad P &= \left(\int_0^n \left[(\gamma(i)^{\frac{\sigma}{\sigma-1}} A(i))^{\frac{1}{1-\beta}} \mu(i) \right]^{1-\frac{1}{\sigma(1-\beta)+\beta}} di \right)^{\frac{\sigma(1-\beta)+\beta}{1-\sigma}}, \\
p(i) &= \gamma(i)^{\frac{\sigma}{\sigma-1}} \left[\left(\int_0^n \left[(\gamma(j)^{\frac{\sigma}{\sigma-1}} A(j))^{\frac{1}{1-\beta}} \mu(j) \right]^{1-\frac{1}{\sigma(1-\beta)+\beta}} dj \right)^{\sigma(1-\beta)+\beta} (\gamma(i)^{\frac{\sigma}{\sigma-1}} A(i))^{\frac{1}{1-\beta}} \mu(i) \right]^{-\frac{1-\beta}{\sigma(1-\beta)+\beta}}, \\
\pi(i) &= (1-\beta) \frac{(\gamma(i)^{\frac{\sigma}{\sigma-1}} A(i))^{\frac{1}{1-\beta}} (1-\frac{1}{\sigma(1-\beta)+\beta}) \mu(i)^{-\frac{1}{\sigma(1-\beta)+\beta}}}{\int_0^n \left[(\gamma(j)^{\frac{\sigma}{\sigma-1}} A(j))^{\frac{1}{1-\beta}} \mu(j) \right]^{1-\frac{1}{\sigma(1-\beta)+\beta}} dj} = \\
&= (1-\beta) \frac{(\gamma(i)^{\frac{\sigma}{\sigma-1}} A(i))^{\frac{1}{1-\beta}} (A(i))^{\frac{\sigma(\sigma-1)}{\sigma(1-\beta)+\beta}} \mu(i)^{-\frac{1}{\sigma(1-\beta)+\beta}}}{\int_0^n \left[(\gamma(j)^{\frac{\sigma}{\sigma-1}} A(j))^{\frac{1}{1-\beta}} \mu(j) \right]^{1-\frac{1}{\sigma(1-\beta)+\beta}} dj},
\end{aligned}$$

the production in sector i is

$$(94) \quad Q(i) = q(i)\mu(i) = \gamma(i)^{-\frac{\sigma}{\sigma-1}} \frac{\left[(\gamma(i)^{\frac{\sigma}{\sigma-1}} A(i))^{\frac{1}{1-\beta}} \mu(i) \right]^{\frac{\sigma(1-\beta)}{\sigma(1-\beta)+\beta}}}{\left(\int_0^n \left[(\gamma(j)^{\frac{\sigma}{\sigma-1}} A(j))^{\frac{1}{1-\beta}} \mu(j) \right]^{1-\frac{1}{\sigma(1-\beta)+\beta}} dj \right)^{\beta}},$$

the income per sector is

$$(95) \quad Y(i) = p(i)Q(i) = \frac{\left[\gamma(i)^{\frac{\sigma}{\sigma-1}} A(i) \mu(i)^{(1-\beta)} \right]^{\frac{\sigma-1}{\sigma(1-\beta)+\beta}}}{\int_0^n \left[(\gamma(j)^{\frac{\sigma}{\sigma-1}} A(j))^{\frac{1}{1-\beta}} \mu(j) \right]^{1-\frac{1}{\sigma(1-\beta)+\beta}} dj},$$

while the number of employed workers per sector is

$$(96) \quad L(i) = \mu(i)l(i) = \frac{\left[\gamma(i)^{\frac{\sigma}{\sigma-1}} A(i) \mu(i)^{\frac{1}{1-\sigma}} \right]^{\frac{\sigma-1}{\sigma(1-\beta)+\beta}}}{\int_0^n \left[(\gamma(j)^{\frac{\sigma}{\sigma-1}} A(j))^{\frac{1}{1-\beta}} \mu(j) \right]^{1-\frac{1}{\sigma(1-\beta)+\beta}} dj}.$$

As before $\Pi = (1-\beta)$.

Proof. Following the same approach as in Section [2.1.1](#), one begins by finding the demand $x(i)$ as a function of $p(i)$ and P . In this case, we have:

$$x(i) = \frac{y}{\left(\frac{p(i)}{\gamma(i)} \right)^{\sigma}} \frac{1}{P^{1-\sigma}}$$

where P is defined as

$$P = \left(\int_0^n \gamma(i)^{\sigma} p(i)^{1-\sigma} di \right)^{1/(1-\sigma)}.$$

On the supply side, things remain as in the symmetric case, and the supply of each firm in the sector i is again given by (59). Thus, the market equilibrium for good i gives, for $Y = y = 1$:

$$\frac{1}{\left(\frac{p(i)}{\gamma(i)}\right)^\sigma} \frac{1}{P^{1-\sigma}} = x(i) = \mu(i)q(i) = \mu(i)A(i)^{1/(1-\beta)}p(i)^{\beta/(1-\beta)}$$

from which, similarly to (66),

$$p(i) = \left[P^{1-\sigma} A(i)^{1/(1-\beta)} \mu(i) \gamma(i)^{-\sigma} \right]^{-(1-\beta)/(\sigma(1-\beta)+\beta)}.$$

Thus, we have:

$$(97) \quad P^{1-\sigma} = \left(\int_0^n \gamma(i)^\sigma p(i)^{1-\sigma} di \right) \\ = P^{-\frac{(1-\sigma)^2(1-\beta)}{\sigma(1-\beta)+\beta}} \left(\int_0^n \gamma(i)^\sigma \left[A(i)^{1/(1-\beta)} \mu(i) \gamma(i)^{-\sigma} \right]^{-(1-\beta)(1-\sigma)/(\sigma(1-\beta)+\beta)} di \right).$$

so that

$$P = \left(\int_0^n \gamma(i)^{\frac{\sigma}{\sigma(1-\beta)+\beta}} \left[A(i)^{1/(1-\beta)} \mu(i) \right]^{1-\frac{1}{\sigma(1-\beta)+\beta}} di \right)^{\frac{\sigma(1-\beta)+\beta}{1-\sigma}} \\ = \left(\int_0^n \left[(\gamma(i)^{\frac{\sigma}{\sigma-1}} A(i))^{\frac{1}{1-\beta}} \mu(i) \right]^{1-\frac{1}{\sigma(1-\beta)+\beta}} di \right)^{\frac{\sigma(1-\beta)+\beta}{1-\sigma}} \\ = \left(\int_0^n \left(\gamma(i)^\sigma A(i)^{\sigma-1} \mu(i)^{(1-\beta)(\sigma-1)} \right)^{\frac{1}{\sigma(1-\beta)+\beta}} di \right)^{\frac{\sigma(1-\beta)+\beta}{1-\sigma}},$$

and thus:

$$(98) \quad p(i) = \left[(P)^{1-\sigma} A(i)^{1/(1-\beta)} \mu(i) \gamma(i)^{-\sigma} \right]^{-(1-\beta)/(\sigma(1-\beta)+\beta)} \\ = \left[\left(\int_0^n \gamma(j)^{\frac{\sigma}{\sigma(1-\beta)+\beta}} \left[A(j)^{1/(1-\beta)} \mu(j) \right]^{1-\frac{1}{\sigma(1-\beta)+\beta}} dj \right)^{\sigma(1-\beta)+\beta} A(i)^{1/(1-\beta)} \mu(i) \gamma(i)^{-\sigma} \right]^{-(1-\beta)/(\sigma(1-\beta)+\beta)} \\ = \gamma(i)^{\frac{\sigma}{\sigma-1}} \left[\left(\int_0^n \left[(\gamma(j)^{\frac{\sigma}{\sigma-1}} A(j))^{\frac{1}{1-\beta}} \mu(j) \right]^{1-\frac{1}{\sigma(1-\beta)+\beta}} dj \right)^{\sigma(1-\beta)+\beta} (\gamma(i)^{\frac{\sigma}{\sigma-1}} A(i))^{\frac{1}{1-\beta}} \mu(i) \right]^{-\frac{1-\beta}{\sigma(1-\beta)+\beta}}.$$

Therefore, we can compute $\pi(i)$,

$$\begin{aligned}
 (99) \quad \pi(i) &= (1 - \beta) A(i)^{1/(1-\beta)} p(i)^{1/(1-\beta)} \\
 &= \frac{(1 - \beta) A(i)^{1/(1-\beta)} [\gamma(i)^{-\sigma} A(i)^{1/(1-\beta)} \mu(i)]^{-1/(\sigma(1-\beta)+\beta)}}{\int_0^n \gamma(j)^{\frac{\sigma}{\sigma(1-\beta)+\beta}} [A(j)^{1/(1-\beta)} \mu(j)]^{-(1-\beta)(1-\sigma)/(\sigma(1-\beta)+\beta)} dj} \\
 &= \frac{(1 - \beta) \gamma(i)^{\frac{\sigma}{\sigma(1-\beta)+\beta}} A(i)^{\frac{1}{1-\beta} \frac{\sigma(1-\beta)+\beta-1}{\sigma(1-\beta)+\beta}} \mu(i)^{-\frac{1}{\sigma(1-\beta)+\beta}}}{\int_0^n \gamma(j)^{\frac{\sigma}{\sigma(1-\beta)+\beta}} A(j)^{\frac{1}{1-\beta} \frac{\sigma(1-\beta)+\beta-1}{\sigma(1-\beta)+\beta}} \mu(j)^{\frac{\sigma(1-\beta)+\beta-1}{\sigma(1-\beta)+\beta}} dj} \\
 &= (1 - \beta) \frac{(\gamma(i)^{\frac{\sigma}{\sigma-1}} A(i))^{\frac{1}{1-\beta}} (1 - \frac{1}{\sigma(1-\beta)+\beta}) \mu(i)^{-\frac{1}{\sigma(1-\beta)+\beta}}}{\int_0^n \left[(\gamma(j)^{\frac{\sigma}{\sigma-1}} A(j))^{\frac{1}{1-\beta}} \mu(j) \right]^{1 - \frac{1}{\sigma(1-\beta)+\beta}} dj}.
 \end{aligned}$$

The other quantities are computed as follows:

$$\begin{aligned}
 (100) \quad Q(i) &= x(i) = \mu(i) (A(i)p(i)^\beta)^{\frac{1}{1-\beta}} \\
 &= \frac{A(i)^{\frac{\sigma}{\sigma(1-\beta)+\beta}} \gamma(i)^{\frac{\beta\sigma}{\sigma(1-\beta)+\beta}} \mu(i)^{1 - \frac{\beta}{\sigma(1-\beta)+\beta}}}{\left(\int_0^n \gamma(j)^{\frac{\sigma}{\sigma(1-\beta)+\beta}} [A(j)^{1/(1-\beta)} \mu(j)]^{1 - \frac{1}{\sigma(1-\beta)+\beta}} dj \right)^\beta} \\
 &= \gamma(i)^{-\frac{\sigma}{\sigma-1}} \frac{\left[(\gamma(i)^{\frac{\sigma}{\sigma-1}} A(i))^{\frac{1}{1-\beta}} \mu(i) \right]^{\frac{\sigma(1-\beta)}{\sigma(1-\beta)+\beta}}}{\left(\int_0^n \left[(\gamma(j)^{\frac{\sigma}{\sigma-1}} A(j))^{\frac{1}{1-\beta}} \mu(j) \right]^{1 - \frac{1}{\sigma(1-\beta)+\beta}} dj \right)^\beta},
 \end{aligned}$$

$$\begin{aligned}
 (101) \quad Y(i) &= p(i)Q(i) \\
 &= A(i)^{\frac{1}{1-\beta}} p(i)^{\frac{1}{1-\beta}} \mu(i) = \frac{A(i)^{\frac{\sigma-1}{\sigma(1-\beta)+\beta}} \gamma(i)^{\frac{\sigma}{\sigma(1-\beta)+\beta}} \mu(i)^{\frac{\sigma(1-\beta)+\beta-1}{\sigma(1-\beta)+\beta}}}{\left(\int_0^n \gamma(j)^{\frac{\sigma}{\sigma(1-\beta)+\beta}} [A(j)^{1/(1-\beta)} \mu(j)]^{1 - \frac{1}{\sigma(1-\beta)+\beta}} dj \right)} \\
 &\quad \frac{[\gamma(i)^{\frac{\sigma}{\sigma-1}} A(i) \mu(i)^{(1-\beta)}]^{\frac{\sigma-1}{\sigma(1-\beta)+\beta}}}{\int_0^n \left[(\gamma(j)^{\frac{\sigma}{\sigma-1}} A(j))^{\frac{1}{1-\beta}} \mu(j) \right]^{1 - \frac{1}{\sigma(1-\beta)+\beta}} dj},
 \end{aligned}$$

$$\begin{aligned}
 (102) \quad L(i) &= \mu(i)l(i) = \frac{A(i)^{\frac{\sigma-1}{\sigma(1-\beta)+\beta}} \gamma(i)^{\frac{\sigma}{\sigma(1-\beta)+\beta}} \mu(i)^{1 - \frac{1}{\sigma(1-\beta)+\beta}}}{\left(\int_0^n \gamma(j)^{\frac{\sigma}{\sigma(1-\beta)+\beta}} [A(j)^{1/(1-\beta)} \mu(j)]^{1 - \frac{1}{\sigma(1-\beta)+\beta}} dj \right)} \\
 &= \frac{[\gamma(i)^{\frac{\sigma}{\sigma-1}} A(i) \mu(i)^{\frac{1}{1-\sigma}}]^{\frac{\sigma-1}{\sigma(1-\beta)+\beta}}}{\int_0^n \left[(\gamma(j)^{\frac{\sigma}{\sigma-1}} A(j))^{\frac{1}{1-\beta}} \mu(j) \right]^{1 - \frac{1}{\sigma(1-\beta)+\beta}} dj}.
 \end{aligned}$$

□

In this case, the arguments from Section 2.2 from (99) we obtain that the distribution equation for μ follows the equation:

$$(103) \quad \begin{cases} \partial_t \mu(t, i) = -(1 - \beta) \frac{\partial_i \left(\mu(t, i) \partial_i \left((\gamma(i))^{\frac{\sigma}{\sigma-1}} A(i) \right)^{\frac{1}{1-\beta}} \left(1 - \frac{1}{\sigma(1-\beta)+\beta} \right) \mu(i)^{-\frac{1}{\sigma(1-\beta)+\beta}} \right)}{\int_0^n \left[(\gamma(j))^{\frac{\sigma}{\sigma-1}} A(j) \right]^{\frac{1}{1-\beta}} \mu(j) dj} \\ \mu(t, 0) = \mu(t, n) \\ \mu(0, i) = \mu_0(i). \end{cases}$$

This is the same equation (21) where $\tilde{A}(i) := \gamma(i)^{\frac{\sigma(1-\beta)}{\sigma(1-\beta)+\beta-1}} A(i)$ appears in place of $A(i)$. Therefore, Theorems 2.3 and 2.5 also apply to this case. The limiting distribution in this case will thus be:

$$\mu^{EQ}(i) = \frac{\tilde{A}(i)^{\frac{\sigma(1-\beta)+\beta-1}{1-\beta}}}{\int_0^n \tilde{A}(j)^{\frac{\sigma(1-\beta)+\beta-1}{1-\beta}} dj} = \frac{\gamma(i)^\sigma A(i)^{\sigma-1}}{\int_0^n \gamma(j)^\sigma A(j)^{\sigma-1} dj}.$$

By substituting this expression into (101) we get the $Y(i)^{EQ}$ described in the main text.

APPENDIX C. THE ECONOMY WITH IMMOBILE WORKERS

Proof of Proposition 3.1. We denoted by L the distribution of labour across different sectors so that in particular, $L(i)$ denotes the mass of workers in the sector i . The demand of labour of *representative firm* of sector i is again:

$$(104) \quad l(i) = \left[\frac{\beta p(i) A(i)}{w(i)} \right]^{1/(1-\beta)} = \left[\frac{\beta p(i) A_0(i) \mu(i)^\eta}{w(i)} \right]^{1/(1-\beta)}$$

the equilibrium wage $w(i)$ is therefore decided by the equality between total demand and total supply of labour, i.e.:

$$(105) \quad l(i) \mu(i) = \left[\frac{\beta p(i) A_0(i) \mu(i)^\eta}{w(i)} \right]^{1/(1-\beta)} \mu(i) = L(i);$$

from which:

$$(106) \quad w(i) = \beta p(i) A_0(i) \mu(i)^\eta \left(\frac{\mu(i)}{L(i)} \right)^{1-\beta}$$

The equilibrium price $p(i)$ is decided by the equality between total demand (see (2)) and total supply, i.e.:

$$(107) \quad x(i) = \frac{y}{p(i)^\sigma} \frac{1}{P^{1-\sigma}} = A_0(i) \mu(i)^\eta \left(\frac{L(i)}{\mu(i)} \right)^\beta \mu(i),$$

i.e.

$$(108) \quad p(i) = \left(\frac{y}{P^{1-\sigma}} \right)^{1/\sigma} (A_0(i)L(i)^\beta \mu(i)^{1+\eta-\beta})^{-1/\sigma}$$

Using (3) and the normalization $Y = y = 1$, from the previous equation, it is also possible to calculate the level of price index in equilibrium:

$$P = \left(\int_0^n p(i)^{1-\sigma} di \right)^{1/(1-\sigma)} = \left(\int_0^n \left(\frac{1}{P^{1-\sigma}} \right)^{(1-\sigma)/\sigma} (A_0(i)L(i)^\beta \mu(i)^{1+\eta-\beta})^{(\sigma-1)/\sigma} di \right)^{1/(1-\sigma)}$$

and then:

$$(109) \quad P = \left(\int_0^n (A_0(i)L(i)^\beta \mu(i)^{1+\eta-\beta})^{1-1/\sigma} di \right)^{\sigma/(1-\sigma)}$$

and

$$(110) \quad p(i) = \frac{(A_0(i)L(i)^\beta \mu(i)^{1+\eta-\beta})^{-1/\sigma}}{\int_0^n (A_0(i)L(i)^\beta \mu(i)^{1+\eta-\beta})^{1-1/\sigma} di}$$

Finally, we can calculate the equilibrium profits per firm:

$$(111) \quad \pi(i) = (1 - \beta) p(i) A_0(i) \mu(i)^\eta \left(\frac{L(i)}{\mu(i)} \right)^\beta = \frac{(1 - \beta) A_0(i)^{1-1/\sigma} L(i)^{\beta(1-1/\sigma)} \mu(i)^{(\eta-\beta)(1-1/\sigma)-1/\sigma}}{\left(\int_0^n (A_0(i)L(i)^\beta \mu(i)^{1+\eta-\beta})^{1-1/\sigma} di \right)}$$

and the equilibrium wage:

$$(112) \quad w(i) = \frac{\beta A_0(i)^{1-1/\sigma} L(i)^{\beta(1-1/\sigma)-1} \mu(i)^{(1+\eta-\beta)(1-1/\sigma)}}{\left(\int_0^n (A_0(i)L(i)^\beta \mu(i)^{1+\eta-\beta})^{1-1/\sigma} di \right)}$$

At sectoral level, we have that the value of production is:

$$(113) \quad Y(i) = \frac{A_0(i)^{1-1/\sigma} L(i)^{\beta(1-1/\sigma)} \mu(i)^{(1+\eta-\beta)(1-1/\sigma)}}{\left(\int_0^n (A_0(i)L(i)^\beta \mu(i)^{1+\eta-\beta})^{1-1/\sigma} di \right)}.$$

The total production in each sector in real terms is

$$Q(i) = Y(i)/p(i) = A_0(i)L(i)^\beta \mu(i)^{1+\eta-\beta}$$

□

REMARK C.1 In particular, using (111) and (113) the following relation holds between $Y(i)$ and $\pi(i)$:

$$(114) \quad Y(i) = \frac{\pi(i)^\varphi [A_0(i)L(i)^\beta]^{(1-\frac{1}{\sigma})(1-\varphi)}}{\int_0^n \pi(j)^\varphi [A_0(j)L(j)^\beta]^{(1-\frac{1}{\sigma})(1-\varphi)} dj}$$

where $\varphi \equiv \frac{(1+\eta-\beta)(\sigma-1)}{(\eta-\beta)(\sigma-1)-1}$.

THEOREM C.2 *Given any strictly positive initial condition $\mu_0(i) \in C^2(S)$, let $A_0, L: S \rightarrow \mathbb{R}$ be strictly positive and in $C^2(S)$ and assume $\frac{1}{\sigma}(1+\eta-\beta) > (\eta-\beta)$. Then Equation (48) has a unique strictly positive solution $\mu(t, \cdot) \in C([0, +\infty), C^2(S))$. The unique solution of Equation (48) can be characterized as the unique gradient flow in $\mathcal{W}_2(S)$ of the functional*

$$\mathcal{F}(\mu) = \left(\frac{\sigma}{\sigma-1} \right) \frac{1-\beta}{1+\eta-\beta} \log \left(\int_0^n [A_0(i)L(i)^\beta \mu(i)^{1+\eta-\beta}]^{\frac{\sigma-1}{\sigma}} di \right) = \frac{(1-\beta)}{(1+\eta-\beta)} \log X(\mu(t)).$$

so that

$$(115) \quad \frac{d\mu}{dt} = \nabla_{\mathcal{W}_2} \mathcal{F}(\mu(t)) = \frac{(1-\beta)}{(1+\eta-\beta)} \frac{\nabla_{\mathcal{W}_2} X(\mu(t))}{X(\mu(t))}, \quad \mu(0) = \mu_0.$$

Moreover, along the equilibrium dynamics, $\mu(t)$ evolves in the direction of $\mathcal{W}_2(S)$ that makes $\mathcal{F}(\mu)$ increase the fastest. In the long-run the distribution of μ converges to the unique steady state μ^{EQ} given by

$$(116) \quad \mu^{EQ}(i) = Y^{EQ}(i) = \frac{[A_0(i)L(i)^\beta]^{\frac{\sigma-1}{(\beta-\eta)(\sigma-1)+1}}}{\int_0^n [A_0(j)L(j)^\beta]^{\frac{\sigma-1}{(\beta-\eta)(\sigma-1)+1}} dj}$$

and

$$(117) \quad \left(\frac{Y(i)}{L(i)} \right)^{EQ} = \frac{A_0(i)^{\frac{\sigma-1}{(\beta-\eta)(\sigma-1)+1}} \cdot L(i)^{\frac{\eta(\sigma-1)-1}{(\beta-\eta)(\sigma-1)+1}}}{\int_0^n [A_0(j)L(j)^\beta]^{\frac{\sigma-1}{(\beta-\eta)(\sigma-1)+1}} dj}.$$

Proof. It uses the same arguments of the proofs of Theorems 2.2, 2.3 and Corollary 2.4. $\square$

Proof of Proposition 3.2. Let us define the exponents for notational simplicity:

$$\gamma_A := \frac{\sigma-1}{1-\eta(\sigma-1)}, \quad \gamma_B := \frac{\sigma-1}{1+(\beta-\eta)(\sigma-1)}.$$

The proposition that $\Delta X^{EQ} \geq 0$ is equivalent to showing that the first term in Equation (53) is greater than or equal to the second. Notice that, due to the conditions for the well-posedness of the economy with perfectly mobile and immobile labour, Equations (33) and (49) respectively, the coefficients γ_A and γ_B have the same signs. The claim can be restated as:

$$(118) \quad \left(\int_0^n A_0(i)^{\gamma_B} L(i)^{\beta\gamma_B} di \right)^{1/\gamma_B} \leq \left(\int_0^n A_0(i)^{\gamma_A} di \right)^{1/\gamma_A}.$$

Assuming first that γ_A and γ_B are positive, raising both sides to the power of γ_B , our goal is to prove:

$$(119) \quad \int_0^n A_0(i)^{\gamma_B} L(i)^{\beta\gamma_B} di \leq \left(\int_0^n A_0(i)^{\gamma_A} di \right)^{\gamma_B/\gamma_A}.$$

We will prove this using Hölder's Inequality. Let us define the conjugate exponents p and q :

$$p = \frac{\gamma_A}{\gamma_B} \quad \text{and} \quad q = \frac{1}{\beta\gamma_B}.$$

These exponents satisfy the condition $\frac{1}{p} + \frac{1}{q} = 1$, as verified by the identity $\gamma_B(1 + \beta\gamma_A) = \gamma_A$.

Applying Hölder's Inequality to the integral on the left-hand side, we have:

$$\begin{aligned} \int_0^n A_0(i)^{\gamma_B} L(i)^{\beta\gamma_B} di &\leq \left(\int_0^n (A_0(i)^{\gamma_B})^p di \right)^{1/p} \left(\int_0^n (L(i)^{\beta\gamma_B})^q di \right)^{1/q} \\ &= \left(\int_0^n A_0(i)^{\gamma_A} di \right)^{\gamma_B/\gamma_A} \left(\int_0^n L(i) di \right)^{\beta\gamma_B} \\ &= \left(\int_0^n A_0(i)^{\gamma_A} di \right)^{\gamma_B/\gamma_A}, \end{aligned}$$

where the last step uses the constraint $\int_0^n L(i) di = 1$. This confirms the inequality and proves that $\Delta X^{EQ} \geq 0$.

Furthermore, equality in Hölder's inequality holds if and only if one function is a constant multiple of the other raised to a specific power. In our case, this condition is $(A_0(i)^{\gamma_B})^p = c \cdot (L(i)^{\beta\gamma_B})^q$ for some constant c . Substituting the expressions for p and q :

$$\begin{aligned} A_0(i)^{\gamma_A} &= c \cdot L(i)^{\beta\gamma_B q} \\ A_0(i)^{\gamma_A} &= c \cdot L(i). \end{aligned}$$

This implies that the minimum occurs for the function $L^*(i)$ that satisfies this condition. To find the constant c , we impose the constraint $\int_0^n L(i) di = 1$:

$$L^*(i) = \frac{1}{c} A_0(i)^{\gamma_A} \implies c = \int_0^n A_0(k)^{\gamma_A} dk.$$

The optimal function is therefore:

$$L^*(i) = \frac{A_0(i)^{\gamma_A}}{\int_0^n A_0(k)^{\gamma_A} dk}.$$

When this optimal function $L^*(i)$ is used, the inequality becomes an equality, which in turn means that the two terms in Equation (53) are equal. Thus, the minimum value of the functional is 0.

To prove the statement in the case where γ_A and γ_B are negative, we first reverse in Equation (119) the direction of the inequality and apply the reverse Hölder inequality. $\square$

APPENDIX D. THE MEANING OF THE GRADIENT IN WASSERSTEIN SPACE ∇_{W_2}

In this appendix, we provide some heuristic remarks and intuition on ∇_{W_2} . For rigorous mathematical details and proofs, see Sections 8.4 and 8.5 of [Ambrosio et al. \(2005\)](#) for the tangent structure and Section 10.4.1 of [Ambrosio et al. \(2005\)](#) and Section 4.5 of [Santambrogio \(2015\)](#) for the explicit structure of the gradient vector field in our case.

The space $\mathcal{W}_2$ consists of probability measures μ , which, to aid intuition, we may think of as having an associated density, also denoted μ throughout the paper. Just as the differential of a function from $\mathbb{R}^n$ to $\mathbb{R}$

is a linear functional representing the infinitesimal change in any direction in $\mathbb{R}^n$, computing the Wasserstein gradient $\nabla_{W_2}\mathcal{F}\mu$ requires defining infinitesimal variations of μ that ensure to remain in W_2 .

The appropriate way to represent such “directions” is via vector fields v on S_1 , assigning to each point in S a tangent vector, i.e., an initial velocity for firms located at that point. Formally, the tangent space at μ is given by:

$$T_\mu\mathcal{W}_2(S) = \overline{\{\nabla\varphi \mid \varphi \in C^\infty(S)\}}^{L^2(S,\mu;\mathbb{R})}.$$

The Wasserstein gradient $\nabla_{W_2}\mathcal{F}(\mu)$ can be identified with a vector field $v_F \in L^2(S,\mu;\mathbb{R})$ such that, for any $v \in T_\mu\mathcal{W}_2(S)$, the gradient acts as:

$$(120) \quad \nabla_{W_2}\mathcal{F}(\mu)[v] = \int_S v_F(i) v(i) d\mu(i).$$

Alternatively, the differential in the direction $v \in T_\mu\mathcal{W}_2(S)$ can be computed via the limit of difference quotients:

$$\nabla_{W_2}\mathcal{F}(\mu)[v] = \lim_{\epsilon \rightarrow 0} \frac{\mathcal{F}((\text{id} + \epsilon v)_\# \mu) - \mathcal{F}(\mu)}{\epsilon} = \left. \frac{d}{d\epsilon} \mathcal{F}((\text{id} + \epsilon v)_\# \mu) \right|_{\epsilon=0},$$

where $(\text{id} + \epsilon v)_\# \mu$ denotes the pushforward measure of μ through the map $x \mapsto x + \epsilon v(x)$. Since the total mass is preserved, transporting μ with velocity v induces the following continuity equation:

$$\partial_t \mu(t, i) = -\partial_i(v(i)\mu(t, i)).$$

Thus, we also have:

$$(121) \quad \begin{aligned} \nabla_{W_2}\mathcal{F}(\mu)[v] &= \lim_{\epsilon \rightarrow 0} \frac{\mathcal{F}((\text{id} + \epsilon v)_\# \mu) - \mathcal{F}(\mu)}{\epsilon} \\ &= \lim_{\epsilon \rightarrow 0} \frac{\mathcal{F}(\mu - \epsilon \partial_i(\mu v)) - \mathcal{F}(\mu)}{\epsilon} = \int_S \frac{\delta \mathcal{F}}{\delta \mu}(i) (-\partial_i(\mu(i)v(i))) di. \end{aligned}$$

Using integration by parts:

$$\int_S \frac{\delta \mathcal{F}}{\delta \mu}(i) (-\partial_i(\mu(i)v(i))) di = \int_S \partial_i \left(\frac{\delta \mathcal{F}}{\delta \mu}(i) \right) \mu(i)v(i) di.$$

Hence:

$$\nabla_{W_2}\mathcal{F}(\mu)[v] = \int_S \mu(i) \partial_i \left(\frac{\delta \mathcal{F}}{\delta \mu}(i) \right) v(i) di.$$

Comparing this with Equation (120), we conclude that the gradient vector field is given by:

$$v_F = \partial_i \left(\frac{\delta \mathcal{F}}{\delta \mu}(i) \right),$$

which is the direction along which the measure evolves under the gradient flow. Therefore, the evolution of μ satisfies the Wasserstein gradient flow equation:

$$\frac{\partial \mu}{\partial t} = -\partial_i(\mu v_F) = -\partial_i \left(\mu \partial_i \left(\frac{\delta \mathcal{F}}{\delta \mu}(i) \right) \right),$$

that is,

$$\nabla_{W_2} \mathcal{F}(\mu) = \partial_i \left(\mu \partial_i \left(\frac{\delta \mathcal{F}}{\delta \mu}(i) \right) \right)$$

as claimed in Footnote [16](#).

APPENDIX E. DIAGNOSTICS OF THE ESTIMATES OF SECTION 4.1

TABLE 1. Estimate of the convergence econometric model for ROE based on 680 four-digit NACE sectors (01–82) over the period 2018–2023 (s.e. in brackets).

	<i>Dependent variable:</i>
	$\Delta ROE(2023)$
$ROE(2018)$	-0.775^{***} (0.023)
Constant	0.105^{***} (0.008)
Observations	680
R^2	0.618
Adjusted R^2	0.617
Residual Std. Error	0.185 (df = 678)
F Statistic	1,096.980 ^{***} (df = 1; 678)
<i>Note:</i>	*p<0.1; **p<0.05; ***p<0.01

Source: our elaboration on ORBIS data.

TABLE 2. Estimate of the convergence econometric model for labour productivity across 680 four-digit NACE sectors (01–82) in 2018 and 2023 (s.e. in brackets).

	<i>Dependent variable:</i>
	$\Delta \log(\text{Labour Productivity}(i, 2023))$
$\log(\text{Labour Productivity}(i, 2018))$	0.003^* (0.001)
Constant	0.027^{***} (0.003)
Observations	679
R^2	0.006
Adjusted R^2	0.004
Residual Std. Error	0.079 (df = 677)
F Statistic	3.819 [*] (df = 1; 677)
<i>Note:</i>	*p<0.1; **p<0.05; ***p<0.01

TABLE 3. Estimate of the growth rate of the value of production for 680 four-digit NACE sectors (01–82) in 2018 and 2023 (s.e. in brackets).

	<i>Dependent variable:</i> $\log(Y(i, 2023)/Y(i, 2018))$
$\log\left(\frac{L(i, t)}{L(i, 0)}\right)$	0.214*** (0.027)
$\log\left(\frac{ROE(i, t)}{ROE(i, 0)}\right)$	0.040*** (0.012)
Constant	0.243*** (0.011)
Observations	612
R ²	0.111
Adjusted R ²	0.108
Residual Std. Error	0.241 (df = 609)
F Statistic	38.000*** (df = 2; 609)
<i>Note:</i>	*p<0.1; **p<0.05; ***p<0.01

Source: our elaboration on ORBIS data.

APPENDIX F. THE IDENTIFICATION OF FIRM DENSITY

As we discussed, unfortunately, we have not a good proxy for the firm density in the sector $\mu(i)$. This also poses a limit to the calculation of good-specific technological parameter $\tilde{A}(i)$. The same consideration can be made for the identification of $\gamma(i)$ in the case of non-symmetric preference. Below, we discuss a possible procedure for the identification of $\mu(i)$ and $\tilde{A}(i)$ based on some assumptions on the share of distribution of $\tilde{A}(i)$ in the economy with labour immobility. The case of non-symmetric preferences is leave to future research.

From Equation (43), we can observe that:

$$(122) \quad \left[\frac{\pi(i)}{1-\beta} \right] \frac{\sigma}{(\eta-\beta)(\sigma-1)-1} L(i) \frac{\beta(1-\sigma)}{(\eta-\beta)(\sigma-1)-1} = \mu(i) \left[\tilde{A}(i) \frac{\sigma-1}{(\eta-\sigma)(\sigma-1)-1} P \frac{\sigma-1}{(\eta-\sigma)(\sigma-1)-1} \right],$$

where the left hand side is observable (given the value of parameters calculated above), while the right hand side is not observable since:

$$(123) \quad P = \left(\int_0^n \left[\tilde{A}(i) L(i)^\beta \mu(i)^{1+\eta-\beta} \right]^{-\frac{1-\sigma}{\sigma}} di \right)^{\frac{\sigma}{1-\sigma}}.$$

The distribution of μ therefore appears as a *coinvolution* of the distribution of π , L and $\tilde{A}$, where the latter is unobservable. A feasible way could be to impose that:

$$(124) \quad \tilde{A}(i) \sim \mathcal{LN}(m_{\tilde{A}}, \sigma_{\tilde{A}}^2)$$

from which

$$(125) \quad \tilde{A}(i) \frac{\sigma-1}{(\eta-\sigma)(\sigma-1)-1} \sim \mathcal{LN} \left(m_{\tilde{A}} \left[\frac{\sigma-1}{(\eta-\sigma)(\sigma-1)-1} \right], \sigma_{\tilde{A}}^2 \left[\frac{\sigma-1}{(\eta-\sigma)(\sigma-1)-1} \right]^2 \right).$$

Therefore, taking the log of both sides of Equation (122):

$$(126) \quad z(i) = y(i) + a(i),$$

where:

$$(127) \quad z(i) \equiv \log \left(\left[\frac{\pi(i)}{1-\beta} \right] \frac{\sigma}{(\eta-\beta)(\sigma-1)-1} L(i) \frac{\beta(1-\sigma)}{(\eta-\beta)(\sigma-1)-1} \right);$$

$$(128) \quad y(i) \equiv \log(\mu(i)); \text{ and}$$

$$(129) \quad a(i) \equiv \log \left(\tilde{A}(i) \frac{\sigma-1}{(\eta-\sigma)(\sigma-1)-1} \right) + \log \left(P \frac{\sigma-1}{(\eta-\sigma)(\sigma-1)-1} \right).$$

Summarizing, we observe z , we assume that

$$(130) \quad a \sim \mathcal{N}(\bar{a}, \sigma_a^2),$$

where $\bar{a}$ and σ_a^2 are to be *consistently* calculated considering that:

$$(131) \quad \bar{a} = \left[\frac{\sigma-1}{(\eta-\sigma)(\sigma-1)-1} \right] \left[m_{\tilde{A}} + \log \left(\left(\int_0^n [\tilde{A}(i) L(i)^\beta \mu(i)^{1+\eta-\beta}]^{-\frac{1-\sigma}{\sigma}} di \right)^{\frac{\sigma}{1-\sigma}} \right) \right],$$

and

$$(132) \quad \sigma_a^2 = \sigma_{\tilde{A}}^2 \left[\frac{\sigma-1}{(\eta-\sigma)(\sigma-1)-1} \right]^2,$$

and we are interested to the distribution of $y \equiv \log(\mu)$.

The Tweedies's Formula (Efron, 2011) gives one possible solution to the estimate of μ starting from Equation (126), stating that:

$$(133) \quad \mathbb{E}[y(i) | z(i)] = z(i) - \bar{a} + \sigma_a^2 \frac{d}{dz} \log f(z)|_{z=z(i)},$$

where $f(z)$ is the marginal density of z . In the estimate, $\bar{a}$ and σ_a^2 are two parameters to be estimated under the constraint that $\exp(y)$ is a marginal distribution, i.e. $\int \exp(y(i)) di = \int \mu(i) di = 1$. This procedure

produces an estimate of $\bar{a}$ and σ_a^2 . From the latter, it is trivial to calculate $\sigma_{\tilde{A}^2}$, i.e.

$$(134) \quad \sigma_{\tilde{A}}^2 = (\sigma_a^2) \frac{(\eta - \sigma)(\sigma - 1) - 1}{\sigma - 1}.$$

Instead, as m_A , we should consider that $\mathbb{E}[a(i) \mid z(i)] = z(i) - \mathbb{E}[y(i) \mid z(i)]$ (indeed, $\mathbb{E}_z[\mathbb{E}[a(i) \mid z(i)]] = \bar{a}$), and therefore:

$$(135) \quad \begin{aligned} \mathbb{E}[\tilde{A}(i)P \mid z(i)] &= \exp\left(\left[\frac{(\eta - \sigma)(\sigma - 1) - 1}{\sigma - 1}\right] \mathbb{E}[a(i) \mid z(i)]\right) = \\ &= \exp\left(\left[\frac{(\eta - \sigma)(\sigma - 1) - 1}{\sigma - 1}\right] (z(i) - \mathbb{E}[y(i) \mid z(i)])\right), \end{aligned}$$

and, finally,

$$(136) \quad \begin{aligned} \int \mathbb{E}[\tilde{A}(i)P \mid z(i)] di &= P \int \mathbb{E}[\tilde{A}(i) \mid z(i)] di = P = \\ &= \int \exp\left(\left[\frac{(\eta - \sigma)(\sigma - 1) - 1}{\sigma - 1}\right] (z(i) - \mathbb{E}[y(i) \mid z(i)])\right) di. \end{aligned}$$

Therefore:

$$(137) \quad \begin{aligned} m_{\tilde{A}} &= \left[\frac{(\eta - \sigma)(\sigma - 1) - 1}{\sigma - 1}\right] \bar{a} - \log P = \\ &= \left[\frac{(\eta - \sigma)(\sigma - 1) - 1}{\sigma - 1}\right] \bar{a} - \log\left(\int \exp\left(\left[\frac{(\eta - \sigma)(\sigma - 1) - 1}{\sigma - 1}\right] (z(i) - \mathbb{E}[y(i) \mid z(i)])\right) di\right), \end{aligned}$$

which complete the characterization of the distribution of $\tilde{A}$. Then, using (126), we get the distribution of μ .

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