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SHARING A COMMON RESOURCE WITH CONCAVE BENEFITS

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Sharing a common resource with concave benefits

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Abstract

A group of agents enjoy concave and single-peak benefit functions from consuming a shared resource. They also value money (transfers). The resource is scarce in the sense that not everybody can consume its peak. The paper characterizes the unique (resource and money) allocation that is efficient, incentive compatible and equal-sharing individual rational. It can be implemented (i) by selling the resource or taxing extraction and redistributing the money collected equally, or (ii) by assigning property rights on equal shares in a competitive market. Next, the paper posits an allocation that is incentive compatible, individual rational and assigns to every agent no more than its peak benefit.

Key words: Common resource, single peak, incentive compatible, no envy, fairness.
JEL codes: D63, Q20

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1 Introduction

Imagine a community of peasants who produce irrigated crops. They have a common access to a flow of water but have different water needs: some might have more land or better irrigation technology than others, or grow different crops. How to share water among them? The answer is obvious if there is enough water to cover their needs: just assign to each peasant what she or he needs. A problem arises when water is scarce in the precise sense that not all needs can be fulfilled. Simple sharing rules like equal sharing might be inefficient because some users might have more than needed. Other sharing rules with unequal shares might be perceived as unfair. It might be difficult to implement when productivity is private information because rationed users would over-report their productivity to obtain more water. Those who are assigned less water might also ask for some sort of monetary compensation, whereas the others should somehow "pay" to consume more.

The paper deals with the problem of sharing a common resource among agents and designing monetary contributions/compensations in a context similar to the above example. The model relies on the following assumptions. First, the agent's benefits from consuming the resource are represented by concave and single-peak functions. Examples include production functions with a shared inputs, e.g. water. Production increases as more input is available until satiation. Marginal productivity is decreasing. It is positive before satiation and becomes negative above this point (e.g. because of water flooding or costly draining).

Second, the agents can be ranked according to both their marginal benefit from consuming the resource up to their peak: those whose value the resource marginally more than others have also equals or higher needs/peak consumption. For instance, they might have more land or grow a crop that requires more water.

Third, the agents might be asked to pay a contribution or to receive a compensation. More precisely, there are two goods in this economy: the resource and money (or production). An allocation specifies how much resource each agent consumes and what he/she pays/receives. It should be feasible, in the sense that what is consumed does not exceed what is available. The transfers must be budget balanced.

Our first concern is to share the resource efficiently in the Pareto sense. It requires to equalize marginal benefits. Our second concern is to implement the efficient resource allocation with imperfect information. Transfers must provide to every agent with incentive to report truthfully benefits (or productivity), no matter what the others do. The allocation should be dominant strategy incentive-compatible (or strategy-proof). The last concern is to insures that everybody accept the allocation. A minimal requirement is individual rationality: each agent should prefer to consume its assigned bundle (resource and money) than nothing. We consider the stronger requirement of equal-sharing individual rationality. The allocation should guarantee to everybody at least the benefit of consuming up to an equal share of the resource.[1]

The paper characterizes an unique allocation that is efficient, (dominant strategy) incentive compatible and equal-sharing individual rational. This allocation can be implemented by selling the resource at a price equal to the marginal benefits and dividing the money collected equally through a lump-sum subsidy. It can also be implemented by assigning property rights on equal shares of the resource providing that the resource is exchanged in a competitive market.

In our framework, incentive-compatibility (or strategy-proofness) is equivalent to no envy (or envy-freeness), which is a central concept in equity theory (Foley, 1967, Varian, 1974, Baumol, 1986). An agent does not envy another agent if its payoff is higher with its assigned consumption bundle (resource and money) than with any other bundle. An allocation satisfies no envy if no agent envies the bundle assigned to another agent. The allocation proposed here is therefore the unique allocation that satisfies no envy, efficiency and equal-sharing individual rationality.

Yet it does not bound upward the agents' payoff. As a consequence, some agents end up gaining more than their benefit from consuming their peak. It is certainly not satisfactory from a fairness point of view: the resource being scarce (in the sense that not everybody

[1]This guaranteed equal split condition is the oldest axiom of the fair division literature, often taken as the definition of fairness (Steinhaus, 1948, Moulin, 1991). Since the equal division of the resource seems a natural solution (but inefficient), an agent could object to an allocation by arguing that it would enjoy a higher benefit with equal sharing (the status-quo). Therefore, every agent should prefer its assigned bundle than consuming (up to) an equal share of the resource.

can consume its peak), by solidarity, it is fair that nobody obtain more than the benefit from consuming this peak. This fairness axiom is called the peak upper bound. It is not compatible with efficiency and no envy (Moulin, 1991). Yet it is compatible with efficiency and the weaker requirement of individual rationality in our framework. The paper posits an allocation that satisfies the peak upper bound, no envy (or incentive-compatibility), efficiency and individual rationality. It can be implemented by selling the resource at the (efficient) marginal benefit price but not redistributing the money collected.

Our model is a two-good version of the fair division problem. This literature has established the compatibility between Pareto efficiency and envy-freeness when preferences are continuous and convex (Varian, 1974, Baumol 1986). Under the assumption of nonsatiated preferences, the equilibrium allocation of a Walrasian exchange economy with equal initial endowments is envy-free and Pareto efficient (in addition to be obviously equal-sharing individual rational) (Foley, 1967, Schmeidler and Vind, 1972, see Thomson and Varian, 1985, for a survey). This paper extends this result to specific non-satiated preferences represented by quasi-linear utility functions concave and single-peak in one good (the resource). It also establishes the uniqueness of such allocation with a large number (a continuum) of small agents.

With single-peak preferences but only one good in the economy (without transfers), Sprumont (1991) has characterized the uniform allocation rule as the unique sharing rule that is efficient, strategy-proof and anonymous.[2] It assigns to each agent the maximum among its peak and a uniform quota (see also Ching, 1992). With two goods, the resource and money, and utility functions concave on the first and linear on the second, the uniform rule is not longer efficient: it fails to equalize marginal benefits. As a consequence, there is room for Pareto improvements through resource re-allocation and transfers.[3]

The rest of the paper proceeds as follow. Section 2 introduces the model and define the axioms. Section 3 characterizes the efficient, incentive compatible and equal-sharing individual rational allocation. It also posits market-based instruments that implements this allocation.

[2]Here, the combination of strategy-proofness and anonymity is equivalent to our definition of incentive-compatibility.

[3]Notice that in our framework the uniform rule is nevertheless incentive compatible and equal-sharing individual rational.

Section 4 points out that some agents might end up with more than their peak benefit. It then provides an allocation that satisfies efficiency, individual rationality and the peak upper bound condition. Section 6 briefly concludes the paper.

2 Model and definitions

Imagine a set of users who share of a common pool resource X. They are labelled according to their "type" θ. Agent θ enjoys a benefit $b(x, \theta)$ for consuming a level x of the resource. $b(., \theta)$ is assumed positive, twice continuously differentiable, strictly concave in x and strictly increasing up to a maximum denoted $\hat{x}_\theta$ for every $\theta \in \Theta$. Agents are ranked according to their productivity parameter θ in the sense that an agent with higher θ has a higher marginal productivity, formally $\frac{\partial^2 b}{\partial\theta\partial x}(x, \theta) > 0$ for every $x \leq \hat{x}_\theta$ and $\theta \in \Theta$. It implies that $\hat{x}_\theta$ is non-decreasing in θ. It is assumed that $b(0, \theta) = 0$ and $\frac{\partial b}{\partial x}(0, \theta) \geq k$ for every $\theta \in \Theta$ with $k > 0$ sufficiently high.[4] The figure 1 below represents an example of benefit functions for three types $\theta' < \theta'' < \theta'''$.

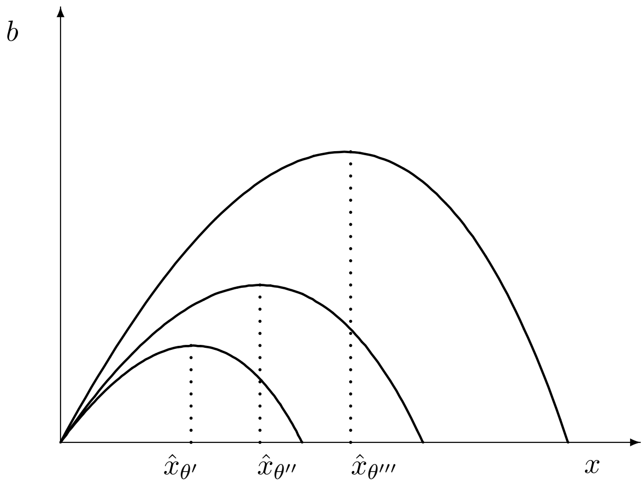

Figure 1

Examples of such benefit functions include the quadratic functions $b(x, \theta) = \theta x - \frac{x^2}{2}$ (with

[4] More precisely, k is assumed higher than the shadow cost of the resource constraint λ (defined later) so that it is efficient to serve everybody. We could have for instance $k = \infty$.

$\underline{\theta}$ high enough) or the log function up to the peak consumption, i.e., $b(x,\theta) = \theta ln(x+1)$ for every $x \leq \hat{x}_\theta$. It could take the form $\theta b(x)$ with b concave, in which case the agents have same peak consumption.

It is common knowledge that θ is distributed according to a density f and a cumulative F in $[\underline{\theta}, \bar{\theta}] \equiv \Theta$, with $f(\theta) > 0$ for every $\theta \in \Theta$.

The resource is scarce in the sense that needs are higher than what is available:

$$\int_\Theta \hat{x}_\theta dF(\theta) > X.$$

An allocation $\{x_\theta, t_\theta\}_{\theta \in \Theta}$ (hereafter denoted $\{x_\theta, t_\theta\}$) is list of bundles (x_θ, t_θ) per type θ. It specifies θ's resource consumption $x_\theta \in \mathbb{R}_+$ and transfer $t_\theta \in \mathbb{R}$ (possibly negative) for every $\theta \in \Theta$. $\{x_\theta\}$ is called a resource allocation whereas $\{t_\theta\}$ is a transfer scheme. It must be *feasible* and *budget balanced* in the sense defined below.

A resource allocation $\{x_\theta\}$ is *feasible* if it satisfies the following resource constraint:

$$\int_\Theta x_\theta dF(\theta) \leq X. \tag{1}$$

A transfer scheme $\{t_\theta\}$ is *budget balanced* if it satisfies the following budget constraint:

$$\int_\Theta t_\theta dF(\theta) \leq 0. \tag{2}$$

The allocation $\{x_\theta, t_\theta\}$ yields to agent θ an utility $u(x_\theta, t_\theta, \theta) = b(x_\theta, \theta) + t_\theta$. Without loss of generality, we restrict attention to resource allocations that satisfy $x_\theta \leq \hat{x}_\theta$ for every $\theta \in \Theta$ because, since the resource is scarce, assigning to any agent θ more than its peak $\hat{x}_\theta$ is inefficient.

The basic axioms are the following.

Definition 1 *The allocation $\{x_\theta, t_\theta\}$ is Incentive Compatible (IC) iff it satisfies the following incentive-compatibility constraints,*

$$b(x_\theta, \theta) + t_\theta \geq b(\min\{x_{\theta'}, \hat{x}_\theta\}, \theta) + t_{\theta'} \text{ for every } \theta' \in \Theta \qquad IC(\theta),$$

for every $\theta \in \Theta$.

An allocation is IC if every agent θ prefers its assigned bundle (x_θ, t_θ) to any other bundle. We assume free disposal: any agent is free not to consumed all the resource. Its consumption level is therefore the maximal value among its peak and its assigned resource level for any report $\theta' \in \Theta$. The above incentive compatibility constraints $IC(\theta)$ (with free disposal) are more stringent than the standard incentive-compatibility constraints (without free disposal).

Definition 2 *The allocation $\{x_\theta, t_\theta\}$ is Individual Rational (IR) iff it satisfies the following individual rationality constraints,*

$$b(x_\theta, \theta) + t_\theta \geq 0 \qquad IR(\theta),$$

for every $\theta \in \Theta$.

An IR allocation assigns a non-negative payoff to every agent.

Definition 3 *The allocation $\{x_\theta, t_\theta\}$ is Equal-Sharing Individual Rational (ESIR) iff it satisfies the following equal-sharing individual rationality constraints,*

$$b(x_\theta, \theta) + t_\theta \geq b(\min\{X, \hat{x}_\theta\}, \theta) \qquad ESIR(\theta),$$

for every $\theta \in \Theta$.

The condition $ESIR(\theta)$ guarantees that agent of type θ obtains at least its benefit with an equal division of the resource under free disposal. To be ESIR, the allocation should assign at least the benefit from consuming an equal share of the resource or less to every agent.

3 The main result

In this transferable utility set-up, efficiency (Pareto optimality) alone selects a unique feasible resource allocation $\{x^*_\theta\}$. It is the resource allocation that maximizes the sum of the benefit functions (e.g. total production) subject to the resource constraint. It solves,

$$\max_{\{x_\theta\}} \int_\Theta b(x_\theta, \theta) dF(\theta) \text{ subject to } \int_\Theta x_\theta dF(\theta) \leq X.$$

Denoting λ the Langragian multiplier associated to the feasibility constraint (1), the efficient allocation satisfies the following first order conditions,

$$\frac{\partial b}{\partial x}(x^*_\theta, \theta) = \lambda \quad \text{for every} \quad \theta \in \Theta \quad \text{with} \quad \lambda > 0. \tag{3}$$

Since $\frac{\partial^2 b}{\partial \theta \partial x} > 0$ for every $x \leq \hat{x}_\theta$, then x^*_θ is increasing in θ.

The efficient resource allocation is unique. It equalizes marginal benefits to the shadow price of the resource. The transfer scheme defined below is the only one that implements the efficient resource allocation in dominant strategy with imperfect information while satisfying equal-sharing individual rationality.

Proposition 1 *The allocation $\{x^*_\theta, t^*_\theta\}$ defined by*

$$\frac{\partial b}{\partial x}(x^*_\theta, \theta) = \lambda \text{ and } t^*_\theta = \lambda(X - x^*_\theta) \text{ for every } \theta \in \Theta,$$

with $\lambda > 0$ and $\int_\Theta x^*_\theta dF(\theta) = X$, is the only (feasible and budget balanced) allocation that is efficient, incentive compatible and equal-sharing individual rational.

The formal proof of Proposition 1 is provided in Appendix. I explain the main argument using the following graphic illustration.

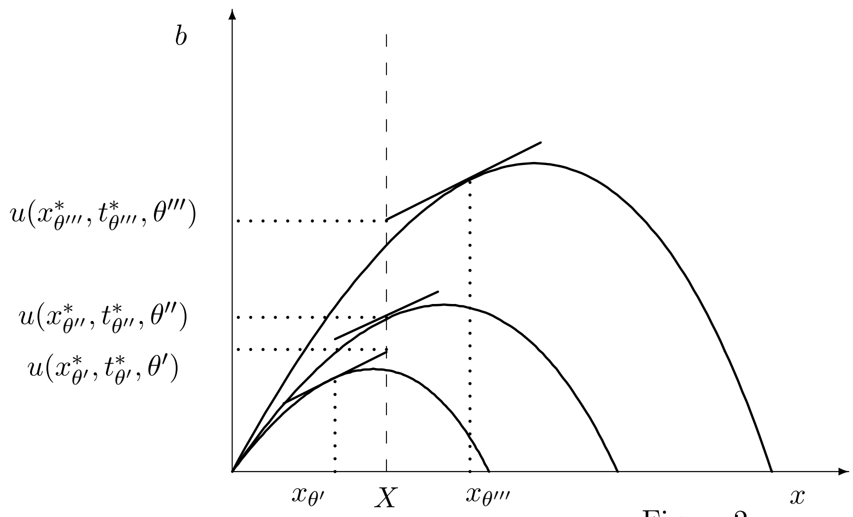

Figure 2

Figure 2 represents the benefit functions of three types of agent $\theta' < \theta'' < \theta'''$. They consume the efficient resource allocation which equalizes marginal benefits to the shadow cost of the resource constraint λ. The tangent of the benefit functions at the efficient resource allocation are all of slope λ. These tangents are represented in Figure 2 for the three types of agents. Agent θ' (resp. θ''') consumes less (resp. more) than the per capita resource X. Agent θ'' consumes exactly X the equal sharing level, which is also the total level of the resource.

An agent θ's utility is $u(x^*_\theta, t^*_\theta, \theta) = b(x^*_\theta, \theta) + \lambda(X - x^*_\theta)$. It is evaluated in Figure 2 at the crossing point of the equal share vertical line X and the tangent to its benefit function at x^*_θ. Agent of type θ' (resp. θ'''), who consumes less (resp. more) than the per capita level X, receives a positive (resp. negative) transfer.

First, by definition, the resource allocation is efficient. Second, it is IC because the selected resource consumption x^*_θ maximizes $b(x, \theta) + \lambda(X - x)$ with respect to x for every $\theta \in \Theta$. Therefore, each agent θ maximizes its utility by choosing its assigned bundle (x^*_θ, t^*_θ). Third, it is ESIR because, due to the concavity of b, the tangent of any agent's benefit function crosses the vertical X line above its benefit at X. Furthermore, when X is lower than the peak consumption $\hat{x}_\theta$, it crosses the vertical line above its benefit at $\hat{x}_\theta$.

Lastly, the proof that no other allocation satisfy Proposition 1's axioms proceed as follow. First, it is shown that first-best resource allocation $\{x^*_\theta\}$ can be implemented with incentive-compatible constraints only if transfers are such that $t_\theta = \gamma - \lambda x^*_\theta$ for every $\theta \in \Theta$, where $\gamma \in \mathbb{R}$ remains to be chosen. On the one hand, the budget balance constraint (2) forces γ to be lower than λX. On the other hand, the ESIR constraint for agent θ'' in Figure 2 (the one who consumes $x^*_{\theta''} = X$) forces γ to be higher than λX.[5] Therefore, the budget balance and equal-sharing individual rationality constraints impose $\gamma = \lambda X$.

The allocation defined in Proposition 1 can be implemented by selling the resource at its shadow price λ and redistributing the money collected λX through a lump-sum subsidy $\sigma \equiv \lambda X$. Each agent consumes the amount that equalizes its marginal benefit to the price, i.e., $\frac{\partial b}{\partial x}(x^*_\theta, \theta) = \lambda$. Agent θ's payoff is $b(x^*_\theta, \theta) - \lambda x^*_\theta + \sigma$. Since $\sigma = \lambda X$, it simplifies to $b(x^*_\theta, \theta) + \lambda(X - x^*_\theta) = u(x^*_\theta, t^*_\theta, \theta)$.

[5] Notice that this agent always exists with a continuum of types $\Theta = [\underline{\theta}, \bar{\theta}]$ and $f(\theta) > 0$ for every $\theta \in \Theta$.

It can also be implemented by dividing the resource if agents can exchange the resource in a competitive market. The resource must then be divided equally: each agent is assigned a property right on a equal share X.[6] The equilibrium price p equals marginal benefits, i.e., $p = \dfrac{\partial b}{\partial x}(x^*_\theta, \theta)$. At this price, any arbitrary agent θ consumes x^*_θ. It sells or buys $X - x^*_\theta$ units of the resource. Its payoff is $b(x^*_\theta, \theta) + p(X - x^*_\theta) = b(x^*_\theta, \theta) + \dfrac{\partial b}{\partial x}(x^*_\theta, \theta)(X - x^*_\theta) = u(x^*_\theta, t^*_\theta, \theta)$.

Before moving to the next section, it is worth to notice that Proposition 1 generalizes to a non-zero amount of money T (positive or negative) to be shared. The transfers are then $t_\theta = \lambda(X - x^*_\theta) + T$ for every $\theta \in \Theta$.[7]

4 The peak upper bound

A fairness principle particularly attractive with single-peak preferences on a scarce resource is the peak upper bound defined below.

Definition 4 *An allocation $\{x_\theta, t_\theta\}$ satisfies the peak upper bound (PUB) iff,*

$$b(x_\theta, \theta) + t_\theta \leq b(\hat{x}_\theta, \theta) \quad PUB(\theta), \tag{4}$$

for every $\theta \in \Theta$.

Unfortunately, the allocation $\{x^*_\theta, t^*_\theta\}$ might violate the peak upper bound. It certainly fails to satisfy $PUB(\theta')$ for the agent θ' represented in Figure 2. It also violates $PUB(\theta)$ for any agent θ whose needs $\hat{x}_\theta$ is less than the equal share X. Indeed, θ's upper bound condition $PUB(\theta)$ with the bundle (x^*_θ, t^*_θ) becomes:

$$\frac{\partial b}{\partial x}(x^*_\theta, \theta)(X - x^*_\theta) \leq b(\hat{x}_\theta, \theta) - b(x^*_\theta, \theta),$$

but if $\hat{x}_\theta \leq X$ then

$$\frac{\partial b}{\partial x}(x^*_\theta, \theta)(X - x^*_\theta) \geq \frac{\partial b}{\partial x}(x^*_\theta, \theta)(\hat{x}_\theta - x^*_\theta) > b(\hat{x}_\theta, \theta) - b(x^*_\theta, \theta),$$

[6] Recall that, with a continuum of agents of mass 1 as assumed here, X is both the total and average level of the resource.

[7] Notice that, since a equal-sharing of T assigns T to everybody, T must be added in the conditions $ESIR(\theta)$ for every $\theta \in \Theta$. Notice also that individual rationality might not hold if T is negative.

where the last inequality is due to the concavity assumption. Furthermore, in the opposite case $\hat{x}_\theta > X$, (x^*_θ, t^*_θ) might also violate $PUB(\theta)$ if $X \geq x^*_\theta + \frac{1}{\lambda}(b(\hat{x}_\theta, \theta) - b(x^*_\theta, \theta))$.

The next proposition posits an allocation that is efficient, incentive compatible, individual rational (but not equal-sharing individual rational) and satisfies the peak upper bound.

Proposition 2 *The allocation* $\{x^*_\theta, t'_\theta\}$ *defined by*

$$\frac{\partial b}{\partial x}(x^*_\theta, \theta) = \lambda \text{ and } t'_\theta = -\lambda x^*_\theta \text{ for every } \theta \in \Theta,$$

with $\lambda > 0$ *and* $\int_\Theta x^*_\theta dF(\theta) = X$, *is efficient, incentive compatible, individual rational and satisfies the peak upper bound.*

The proof, relegated in Appendix, is illustrated in Figure 3 below.

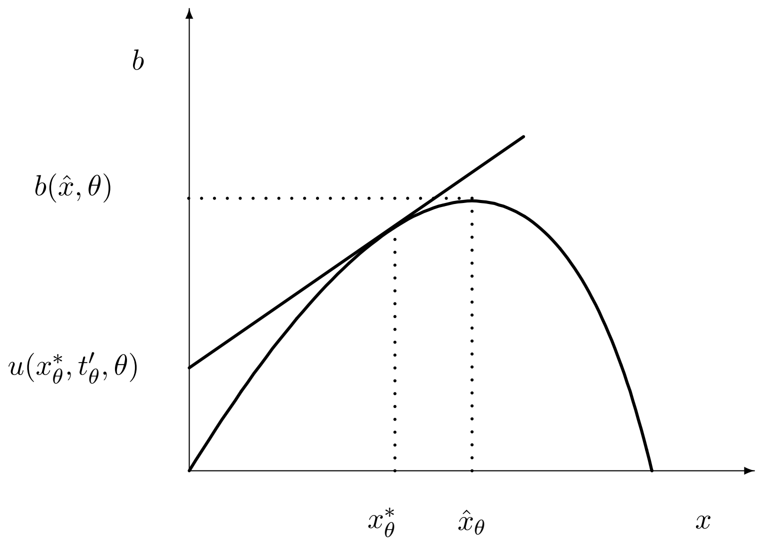

Figure 3

Agent θ's utility $u(x^*_\theta, t'_\theta, \theta)$ is evaluated where the tangent to θ's benefit function at the efficient level x^*_θ (of slope λ) crosses the vertical axe. Due to the concavity of b, it is located above 0 but below the peak consumption $\hat{x}_\theta$ so long as $\lambda > 0$, i.e. the resource is scarce.

The allocation $\{x^*_\theta, t'_\theta\}$ can be implemented by selling the resource at $\lambda = \frac{\partial b}{\partial x}(x^*_\theta, \theta)$ but without redistributing the money collected. Yet some redistribution might be allowed while satisfying the peak upper bound. The lump-sum subsidy σ must then be such that:

$$\sigma \leq \min_{\theta \in \Theta} b(\hat{x}_\theta, \theta) - b(x^*_\theta, \theta) + \lambda x^*_\theta, \tag{5}$$

Differentiating the above right-hand side with respect to θ and using the envelop theorem shows that the right hand side in (5) is increasing in θ if $\frac{\partial b}{\partial \theta}(\hat{x}_\theta, \theta) \geq \frac{\partial b}{\partial \theta}(x^*_\theta, \theta)$ for every $\theta \in \Theta$.[8] In this case, the subsidy is constraint by the lower type $\underline{\theta}$'s benefit. The highest per capita subsidy is then $\sigma = b(\hat{x}_\theta, \underline{\theta}) - b(x^*_{\underline{\theta}}, \underline{\theta}) + \lambda x^*_{\underline{\theta}}$.

5 Conclusion

This paper considers a single resource division problem with concave, single-peak and single-crossing benefit functions and monetary transfers. Its main result is that only one division of the resource and vector of transfers is Pareto optimal, envy-free (incentive compatibility) and guarantee to every agent a payoff not lower than with an equal split of the resource. This allocation is characterized. It can be implemented with simple market-based instruments (taxation or privatization). It might be perceived as unfair because it assigns to some agents more than their highest benefit. Another allocation satisfies Pareto optimality and assigns to every agent not more than its highest benefit but not less than zero. It requires to sell the resource at its shadow price.

[8]It holds for instance for $b(x, \theta) = \theta b(x)$ or $b(x, \theta) = \theta x - \frac{x^2}{2}$.

A Proof of Proposition 1

<u>Part 1</u>: *The allocation $\{x^*_\theta, t_\theta\}$ is budget balanced, efficient, IC and ESIR.*

First, the resource allocation $\{x^*_\theta\}$ is efficient by definition.

Second, since $\int_\Theta t_\theta dF(\theta) = \lambda X - \lambda \int_\Theta x^*_\theta dF(\theta) = 0$, where the last equality is due to the binding resource constraint, the transfer scheme is budget balanced.

Third, I show that the allocation $\{x^*_\theta, t^*_\theta\}$ is ESIR. The constraint $ESIR(\theta)$ writes

$$b(x^*_\theta, \theta) - \lambda x^*_\theta + \lambda X \geq b(\min\{X, \hat{x}_\theta\}, \theta). \tag{6}$$

The proof relies on the concavity of the benefit function. Consider all $\theta \in \Theta$ such that $x^*_\theta \geq X$. Since b is concave,

$$b(x^*_\theta, \theta) - b(X, \theta) \geq \frac{\partial b}{\partial x}(x^*_\theta, \theta)(x^*_\theta - X).$$

Substitute $\frac{\partial b}{\partial x}(x^*_\theta, \theta)$ with λ to obtain:

$$b(x^*_\theta, \theta) + \lambda(X - x^*_\theta) \geq b(X, \theta),$$

which is (6) when $x^*_\theta \leq X \leq x^*_\theta \leq \hat{x}_\theta$.

Consider now all users with $x^*_\theta \leq X \leq \hat{x}_\theta$. Since b is concave,

$$b(X, \theta) - b(x^*_\theta, \theta) \leq \frac{\partial b}{\partial x}(x^*_\theta, \theta)(X - x^*_\theta).$$

Replace $\frac{\partial b}{\partial x}(x^*_\theta, \theta)$ by λ to obtain:

$$b(X, \theta) \leq b(x^*_\theta, \theta) + \lambda(X - x^*_\theta),$$

which is (6) for $X \leq \hat{x}_\theta$. For users θ such that $x^*_\theta \leq X$ but $X \geq \hat{x}_\theta$, the equal sharing benefit is $b(\hat{x}_\theta, \theta)$, i.e. the peak benefit. Again, b concave and $x^*_\theta \leq \hat{x}_\theta$ imply:

$$b(\hat{x}_\theta, \theta) - b(x^*_\theta, \theta) \leq \frac{\partial b}{\partial x}(x^*_\theta, \theta)(\hat{x}_\theta - x^*_\theta),$$

for every $\theta \in \Theta$, which is equivalent to,

$$b(x^*_\theta, \theta) + \lambda(X - x^*_\theta) \geq b(\hat{x}_\theta, \theta) + \lambda(X - \hat{x}_\theta). \tag{7}$$

For $\hat{x}_\theta \leq X$, the above condition implies:

$$b(x^*_\theta, \theta) + \lambda(X - x^*_\theta) \geq b(\hat{x}_\theta, \theta),$$

which is constraint (6) for any $\theta \in \Theta$ such that $\hat{x}_\theta \leq X$.

Fourth, I turn to the IC condition. Pick any agent θ. If it selects the bundle $(x_{\hat{\theta}}, t_{\hat{\theta}})$, it obtains $u(x^*_\theta, t^*_\theta, \theta) = b(x_{\hat{\theta}}, \theta) + \lambda(X - x_{\hat{\theta}})$. The condition $\dfrac{\partial b}{\partial x}(x^*_\theta, \theta) = \lambda$ is the first-order condition of the maximization program $\max_x b(x, \theta) - \lambda x + \lambda X$. Therefore, $b(x^*_\theta, \theta) + \lambda(X - x^*_\theta) \geq b(x^*_{\theta'}, \theta) + \lambda(X - x^*_{\theta'})$ for every $\theta' \in \Theta$. Replace $\lambda(X - x^*_\theta)$ and $\lambda(X - x^*_{\theta'})$ by, respectively, t^*_θ and $t^*_{\theta'}$ to obtain

$$b(x^*_\theta, \theta) + t^*_\theta \geq b(x_{\theta'}, \theta) + t_{\theta'} \tag{8}$$

for every $\theta' \in \Theta$, which is the incentive-compatibility condition *without* free disposal. Condition (7) implies $b(x^*_\theta, \theta) + \lambda(X - x^*_\theta) \geq b(\hat{x}_\theta, \theta) + \lambda(X - x_{\theta'})$ for every $x_{\theta'} \geq \hat{x}_\theta$. Since $t_{\theta'} = \lambda(X - x_{\theta'})$ and $t^*_\theta = \lambda(X - x_{\theta'})$, it leads to

$$b(x^*_\theta, \theta) + t^*_\theta - x^*_\theta \geq b(\hat{x}_\theta, \theta) + t_{\theta'},$$

for every θ' such that $x_{\theta'} \geq \hat{x}_\theta$. The last conditions and inequality (8) for every $\theta' \in \Theta$ imply $b(x^*_\theta, \theta) + t^* \geq b(\min\{x^*_{\theta'}, \hat{x}_{\theta'}\}, \theta) + t_{\theta'}$, i.e. $IC(\theta)$. Since the arguments applies for every $\theta \in \Theta$, the allocation $\{x^*_\theta, t^*_\theta\}$ is IC.

<u>Part 2</u>: Budget balance, efficiency, feasibility, equal-sharing individual rationality and incentive-compatibility imply $x_\theta = x^*_\theta$ and $t_\theta = \lambda(X - x^*_\theta)$ for every $\theta \in \Theta$.

We already know that efficiency implies $\dfrac{\partial b}{\partial x}(x^*_\theta, \theta) = \lambda$, where λ is the Lagrangian multiplier associated with the above maximization program.

Now, for every $\theta \in \Theta$, the first-order condition of the maximization program defined in $IC(\theta)$ yields,

$$\frac{\partial b}{\partial x}(x^*_\theta, \theta)\frac{dx^*_\theta}{d\theta} + \frac{dt_\theta}{d\theta} = 0.$$

Substitute $\dfrac{\partial b}{\partial x}(x^*_\theta, \theta) = \lambda$ to obtain,

$$\frac{dt_\theta}{d\theta} = -\lambda\frac{dx^*_\theta}{d\theta}.$$

The above equality implies that $\exists\gamma$ such that $t_\theta = -\lambda x^*_\theta + \gamma$. Hence, we have shown that $IC(\theta)$ implies $t_\theta = \gamma - \lambda x^*_\theta$ for every $\theta \in \Theta$ for some $\gamma \in \mathbb{R}$.

The budget balance constraint writes:

$$\gamma - \lambda \int_{\Theta} x^* dF(\theta) \leq 0.$$

Due to the feasibility constraint, it implies $\gamma \leq \lambda X$.

The ESIR constraint writes:

$$b(x_\theta^*, \theta) - \lambda x_\theta^* + \gamma \geq b(X, \theta),$$

for every $\theta \in \Theta$.

Now it is easy to show that $\exists \tilde{\theta}$ such that $x_{\tilde{\theta}}^* = X$. Indeed if $\exists \theta'$ such that $x_{\theta'}^* < X$ then $\exists \theta''$ such that $x_{\theta''}^* > X$ (and vice-versa). Since x_θ^* is increasing and continuous in θ (as shown by differentiating the efficiency first-order condition), then there exists $\tilde{\theta} \in (\theta', \theta'')$ such that $x_{\tilde{\theta}}^* = X$. For $\tilde{\theta}$ such that $x_{\tilde{\theta}}^* = X$, the ESIR constraint writes

$$b(X, \theta) - \lambda X + \gamma \geq b(X, \theta),$$

which leads to $\gamma \geq \lambda X$. Hence, combining budget balance, which implies $\gamma \leq \lambda X$, with ESIR, which implies $\gamma \geq \lambda X$, shows that $\gamma = \lambda X$. Therefore $t_\theta = \lambda(X - x_\theta^*)$ for every $\theta \in \Theta$.

B Proof of Proposition 2

First, the allocation satisfies incentive-compatibility for the same reason as before. Second, it is budget balanced because transfers are negative. Third, it yields to any agent θ an utility level,

$$u(x_\theta^*, t'_\theta, \theta) = b(x_\theta^*, \theta) - \frac{\partial b}{\partial x}(x_\theta^*, \theta) x_\theta^*.$$

Since $x_\theta^* < \hat{x}_\theta$, $PUB(\theta)$ is obviously satisfied for every $\theta \in \Theta$. Furthermore, the concavity of b implies $b(x_\theta^*, \theta) - b(0, \theta) > \frac{\partial b}{\partial x}(x_\theta^*, \theta) x_\theta^*$. Since $b(0, \theta) = 0$ for every $\theta \in \Theta$, then $b(x_\theta^*, \theta) > \frac{\partial b}{\partial x}(x_\theta^*, \theta) x_\theta^*$ which implies that every agent θ's payoff is positive. Hence, individual rationality holds for every $\theta \in \Theta$.

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